31.18

$$\left(2+\sqrt{3}\right)^x+\left(2-\sqrt{3}\right)^x=14$$

Έστω $\quad 2+\sqrt{3}=\alpha \quad , \quad 2-\sqrt{3}=\beta$

Ισχύει $\quad \left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)=2^2-\sqrt{3}^2=1 \Rightarrow \alpha\cdot\beta=1 \Rightarrow \beta=\dfrac{1}{\alpha}$

Άρα η εξίσωση γράφεται :

$\alpha^x+\beta^x=14 \Rightarrow \alpha^x+\dfrac{1}{\alpha^x}=14$ $\quad$ έστω $\quad \alpha^x=\omega>0$ $\quad$ οπότε :

$$\omega+\frac{1}{\omega}=14 \Rightarrow \omega^2-14\omega+1=0$$

$$\Delta=192=64\cdot 3 \qquad \omega_{1,2}=\frac{14\pm 8\sqrt{3}}{2}=7\pm 4\sqrt{3}\overset{\substack{\left(2+\sqrt{3}\right)^2=4+4\sqrt{3}+3\\ \left(2-\sqrt{3}\right)^2=4-4\sqrt{3}+3}}{=}\left(2\pm\sqrt{3}\right)^2$$

Άρα

$\omega=\left(2+\sqrt{3}\right)^2$

$\Rightarrow \alpha^x=\left(2+\sqrt{3}\right)^2$

$\Rightarrow \left(2+\sqrt{3}\right)^x=\left(2+\sqrt{3}\right)^2$

$\Rightarrow x=2$

ή

$\omega=\left(2-\sqrt{3}\right)^2$

$\Rightarrow \left(2+\sqrt{3}\right)^x=\left(2-\sqrt{3}\right)^2$

$\overset{2-\sqrt{3}=\frac{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)}{2+\sqrt{3}}=\frac{1}{2+\sqrt{3}}}{\Rightarrow} \left(2+\sqrt{3}\right)^x=\dfrac{1}{\left(2+\sqrt{3}\right)^2}$

$\Rightarrow x=-2$