6.Δ1

α) Είναι $|\overrightarrow{ΒΓ}|^2 = \overrightarrow{ΒΓ}^2 = (\overrightarrow{ΑΓ} - \overrightarrow{ΑΒ})^2 = \overrightarrow{ΑΓ}^2 - 2\overrightarrow{ΑΓ}\cdot\overrightarrow{ΑΒ} + \overrightarrow{ΑΒ}^2 =$

$$= |\overrightarrow{ΑΓ}|^2 - 2|\overrightarrow{ΑΒ}|\cdot|\overrightarrow{ΑΓ}|\cdot συν(\widehat{\overrightarrow{ΑΒ}, \overrightarrow{ΑΓ}}) + |\overrightarrow{ΑΒ}|^2 =$$

$$= 4^2 - 2\cdot 2\cdot 4\cdot συν\frac{\pi}{3} + 2^2 = 16 - 8 + 4 = 12$$

και $\overrightarrow{ΑΔ} = \overrightarrow{ΑΒ} + \overrightarrow{ΒΔ} = \overrightarrow{ΑΒ} + \dfrac{2\overrightarrow{ΒΓ}}{3} = \overrightarrow{ΑΒ} + \dfrac{2(\overrightarrow{ΑΓ} - \overrightarrow{ΑΒ})}{3} = \dfrac{\overrightarrow{ΑΒ} + 2\overrightarrow{ΑΓ}}{3}$

Άρα $|\overrightarrow{ΑΔ}|^2 = (\overrightarrow{ΑΔ})^2 = \dfrac{(\overrightarrow{ΑΒ} + 2\overrightarrow{ΑΓ})^2}{9} = \dfrac{\overrightarrow{ΑΒ}^2 + 4\overrightarrow{ΑΒ}\cdot\overrightarrow{ΑΓ} + 4\overrightarrow{ΑΓ}^2}{9} =$

$$= \frac{|\overrightarrow{ΑΒ}|^2 + 4|\overrightarrow{ΑΒ}|\cdot|\overrightarrow{ΑΓ}|\cdot συν(\widehat{\overrightarrow{ΑΒ}, \overrightarrow{ΑΓ}}) + 4|\overrightarrow{ΑΓ}|^2}{9} =$$

$$= \frac{2^2 + 4\cdot 2\cdot 4\cdot\frac{1}{2} + 4\cdot 4^2}{9} = \frac{84}{9} = \frac{4\cdot 21}{9}$$

Άρα $|\overrightarrow{ΑΔ}|^2 = \dfrac{4\cdot 21}{9} \Rightarrow |\overrightarrow{ΑΔ}| = \dfrac{2\sqrt{21}}{3}$

β) Είναι $\overrightarrow{ΑΔ} = \overrightarrow{ΑΒ} + \overrightarrow{ΒΔ} \Rightarrow \overrightarrow{ΒΔ} = \overrightarrow{ΑΔ} - \overrightarrow{ΑΒ} \Rightarrow \overrightarrow{ΒΔ}^2 = (\overrightarrow{ΑΔ} - \overrightarrow{ΑΒ})^2 \Rightarrow$

$$\Rightarrow |\overrightarrow{ΒΔ}|^2 = |\overrightarrow{ΑΔ}|^2 - 2\overrightarrow{ΑΔ}\cdot\overrightarrow{ΑΒ} + |\overrightarrow{ΑΒ}|^2 \Rightarrow$$

$$\Rightarrow \left(\frac{2\overrightarrow{ΒΓ}}{3}\right)^2 = |\overrightarrow{ΑΔ}|^2 - 2\overrightarrow{ΑΔ}\cdot\overrightarrow{ΑΒ} + |\overrightarrow{ΑΒ}|^2 \Rightarrow$$

$$\Rightarrow \frac{4}{9}\cdot|\overrightarrow{ΒΓ}|^2 = |\overrightarrow{ΑΔ}|^2 - 2\overrightarrow{ΑΔ}\cdot\overrightarrow{ΑΒ} + |\overrightarrow{ΑΒ}|^2 \Rightarrow$$

$$\Rightarrow \frac{4}{9}\cdot 12 = \left(\frac{2\sqrt{21}}{3}\right)^2 - 2\overrightarrow{ΑΔ}\cdot\overrightarrow{ΑΒ} + 2^2 \Rightarrow \overrightarrow{ΑΔ}\cdot\overrightarrow{ΑΒ} = 4$$

Άρα $συνω = συν(\widehat{\overrightarrow{ΑΔ}, \overrightarrow{Αβ}}) = \dfrac{\overrightarrow{ΑΔ}\cdot\overrightarrow{ΑΒ}}{|\overrightarrow{ΑΔ}|\cdot|\overrightarrow{Αβ}|} = \dfrac{4}{\frac{2\sqrt{21}}{3}\cdot 2} = \dfrac{3}{\sqrt{21}} = \dfrac{\sqrt{21}\cdot 3}{21} = \dfrac{\sqrt{21}}{7}$

6.Δ2

α) Είναι $\overrightarrow{ΜΑ}\cdot\overrightarrow{ΜΒ} = \dfrac{\overrightarrow{ΔΑ}}{2}\cdot(\overrightarrow{ΜΑ} + \overrightarrow{ΑΒ}) = \dfrac{\overrightarrow{ΔΑ}}{2}\cdot\overrightarrow{ΜΑ} + \dfrac{\overrightarrow{ΔΑ}}{2}\cdot\overrightarrow{ΑΒ} =$

$= \dfrac{\overrightarrow{ΔΑ}^2}{4} + 0 = \dfrac{\overrightarrow{ΔΑ}^2}{4}$ αφού $\overrightarrow{ΔΑ} \perp \overrightarrow{ΑΒ} \Rightarrow \overrightarrow{ΔΑ}\cdot\overrightarrow{ΑΒ} = 0$

και $\overrightarrow{ΜΓ}\cdot\overrightarrow{ΜΔ} = (\overrightarrow{ΜΔ} + \overrightarrow{ΔΓ})\cdot\overrightarrow{ΜΔ} = \left(\dfrac{\overrightarrow{ΑΔ}}{2} + \overrightarrow{ΔΓ}\right)\cdot\dfrac{\overrightarrow{ΑΔ}}{2} =$

$= \dfrac{\overrightarrow{ΔΑ}^2}{4} + 0 = \dfrac{\overrightarrow{ΔΑ}^2}{4}$ αφού $\overrightarrow{ΑΔ} \perp \overrightarrow{ΔΓ} \Rightarrow \overrightarrow{ΑΔ}\cdot\overrightarrow{ΔΓ} = 0$

β) Έχουμε

$$\overrightarrow{ME}\cdot\overrightarrow{B\Gamma} \overset{\substack{\overrightarrow{ME}=\overrightarrow{MA}+\overrightarrow{AE} \\ \overrightarrow{B\Gamma}=\overrightarrow{BM}+\overrightarrow{M\Gamma}}}{=} \left(\overrightarrow{MA}+\overrightarrow{AE}\right)\cdot\left(\overrightarrow{BM}+\overrightarrow{M\Gamma}\right)=\overrightarrow{MA}\cdot\overrightarrow{BM}+\overrightarrow{MA}\cdot\overrightarrow{M\Gamma}+\underbrace{\cancel{\overrightarrow{AE}\cdot\overrightarrow{BM}}}_{\overrightarrow{AE}\perp\overrightarrow{BM}}+\overrightarrow{AE}\cdot\overrightarrow{M\Gamma}$$

$$\overset{\overrightarrow{MA}\cdot\overrightarrow{BM}=-\overrightarrow{MA}\cdot\overrightarrow{MB}\overset{\alpha)\ \text{ερώτημα}}{=}-\overrightarrow{M\Gamma}\cdot\overrightarrow{M\Delta}}{=} -\overrightarrow{M\Gamma}\cdot\overrightarrow{M\Delta}+\overrightarrow{MA}\cdot\overrightarrow{M\Gamma}+\overrightarrow{AE}\cdot\overrightarrow{M\Gamma}=\overrightarrow{M\Gamma}\left(-\overrightarrow{M\Delta}+\overrightarrow{MA}+\overrightarrow{AE}\right)=$$

$$=\overrightarrow{M\Gamma}\left(\overrightarrow{\Delta M}+\overrightarrow{MA}+\overrightarrow{AE}\right)=\overrightarrow{M\Gamma}\cdot\overrightarrow{\Delta E}\overset{\overrightarrow{M\Gamma}\perp\overrightarrow{\Delta E}}{=}0$$

Οπότε $\overrightarrow{ME}\cdot\overrightarrow{M\Gamma}=0\Rightarrow\overrightarrow{ME}\perp\overrightarrow{M\Gamma}$

6.Δ3

Έστω $A(0,0)$, $B(\alpha,0)$, $\Gamma(\alpha,\alpha)$, $\Delta(0,\alpha)$

και $M(\beta,\alpha)$, $N(0,\beta)$, $K(\alpha-\beta,0)$, $\Lambda(\alpha,\alpha-\beta)$

Οπότε έχουμε $\overrightarrow{MN}=(0-\beta,\beta-\alpha)=(-\beta,\beta-\alpha)$

και $\overrightarrow{\Lambda K}=(\alpha-\beta-0,\beta-\alpha)=(-\beta,\beta-\alpha)$

Οπότε $\overrightarrow{MN}=\overrightarrow{\Lambda K}$ (1)

Επίσης είναι $\overrightarrow{M\Lambda}=(\alpha-\beta,\alpha-\beta-\alpha)=(\alpha-\beta,-\beta)$

και $\overrightarrow{NK}=(\alpha-\beta-0,0-\beta)=(\alpha-\beta,-\beta)$

Οπότε $\overrightarrow{M\Lambda}=\overrightarrow{NK}$ (2)

Από (1) , (2) προκύπτει ότι το ΚΛΜΝ είναι τετράγωνο

6.Δ4

Έστω $A(0,\alpha)$, $B(\alpha,\alpha)$, $\Gamma(\alpha,0)$, $\Delta(0,0)$

και $E(0,2\alpha)$, $Z(2\alpha,\alpha)$, $H(\alpha,-\alpha)$, $\Theta(-\alpha,0)$

Οπότε έχουμε $\overrightarrow{EZ}=(2\alpha-0,\alpha-2\alpha)=(2\alpha,-\alpha)$

και $\overrightarrow{\Theta H}=(\alpha-(-\alpha),-\alpha-0)=(2\alpha,-\alpha)$

Οπότε $\overrightarrow{EZ}=\overrightarrow{\Theta H}$ (1)

Επίσης είναι $\overrightarrow{E\Theta}=(-\alpha-0,0-2\alpha)=(-\alpha,-2\alpha)$

και $\overrightarrow{ZH}=(\alpha-2\alpha,-\alpha-\alpha)=(-\alpha,-2\alpha)$

Οπότε $\overrightarrow{E\Theta}=\overrightarrow{ZH}$ (2)

Από (1) , (2) προκύπτει ότι το ΚΛΜΝ είναι τετράγωνο

6.Δ5

Έστω $A(0,0)$, $B(\alpha,0)$, $\Gamma(\alpha,\alpha)$, $\Delta(0,\alpha)$ και $K(0,\beta)$, $\Lambda(\beta,\alpha)$

Τα Μ , Ν είναι μέσα των ΚΛ και ΒΛ, άρα έχουν συντεταγμένες

$$M\left(\frac{0+\beta}{2},\frac{\alpha+\beta}{2}\right)\Rightarrow M\left(\frac{\beta}{2},\frac{\alpha+\beta}{2}\right)$$

$$N\left(\frac{\beta+\alpha}{2},\frac{\alpha+0}{2}\right)\Rightarrow N\left(\frac{\alpha+\beta}{2},\frac{\alpha}{2}\right)$$

Οπότε έχουμε $\overrightarrow{MN}=\left(\frac{\alpha+\beta}{2}-\frac{\beta}{2},\frac{\alpha}{2}-\frac{\alpha+\beta}{2}\right)\Rightarrow$

$\Rightarrow\overrightarrow{MN}=\left(\frac{\alpha}{2},-\frac{\beta}{2}\right)$ και $\overrightarrow{A\Lambda}=(\beta-0,\alpha-0)=(\beta,\alpha)$

Οπότε τελικά $\overrightarrow{MN}\cdot\overrightarrow{A\Lambda}=\frac{\alpha}{2}\cdot\beta-\frac{\beta}{2}\cdot\alpha=0\Rightarrow\overrightarrow{MN}\perp\overrightarrow{A\Lambda}$

6.Δ6

Είναι $\left|\vec{\alpha}+\vec{\beta}\right|+\left|\vec{\alpha}-\vec{\beta}\right|=6\Rightarrow\left|\vec{\alpha}+\vec{\beta}\right|=6-\left|\vec{\alpha}-\vec{\beta}\right|\Rightarrow\left|\vec{\alpha}+\vec{\beta}\right|^2=\left(6-\left|\vec{\alpha}-\vec{\beta}\right|\right)^2\Rightarrow$

$$\Rightarrow\vec{\alpha}^2+2\vec{\alpha}\cdot\vec{\beta}+\vec{\beta}^2=36-12\left|\vec{\alpha}-\vec{\beta}\right|+\left|\vec{\alpha}-\vec{\beta}\right|^2\Rightarrow$$

$$\Rightarrow 9+2\vec{\alpha}\cdot\vec{\beta}=36-12\left|\vec{\alpha}-\vec{\beta}\right|+\vec{\alpha}^2-2\vec{\alpha}\cdot\vec{\beta}+\vec{\beta}^2\Rightarrow$$

$$\Rightarrow 9+2\vec{\alpha}\cdot\vec{\beta}=36-12\left|\vec{\alpha}-\vec{\beta}\right|+9-2\vec{\alpha}\cdot\vec{\beta}\Rightarrow$$

$$\Rightarrow 12\left|\vec{\alpha}-\vec{\beta}\right|=36-4\vec{\alpha}\cdot\vec{\beta}\Rightarrow 3\left|\vec{\alpha}-\vec{\beta}\right|=9-\vec{\alpha}\cdot\vec{\beta}\Rightarrow$$

$$\Rightarrow 9\left(\vec{\alpha}^2-2\vec{\alpha}\cdot\vec{\beta}+\vec{\beta}^2\right)=81-18\vec{\alpha}\cdot\vec{\beta}+\left(\vec{\alpha}\cdot\vec{\beta}\right)^2\Rightarrow$$

$$\Rightarrow 9\left(9-2\vec{\alpha}\cdot\vec{\beta}\right)=81-18\vec{\alpha}\cdot\vec{\beta}+\left(\vec{\alpha}\cdot\vec{\beta}\right)^2\Rightarrow$$

$$\Rightarrow 81-18\vec{\alpha}\cdot\vec{\beta}=81-18\vec{\alpha}\cdot\vec{\beta}+\left(\vec{\alpha}\cdot\vec{\beta}\right)^2\Rightarrow$$

$$\Rightarrow\left(\vec{\alpha}\cdot\vec{\beta}\right)^2=0\Rightarrow\vec{\alpha}\cdot\vec{\beta}=0\Rightarrow\vec{\alpha}\perp\vec{\beta}$$

6.Δ7

Είναι $\vec{\alpha}_1+\vec{\alpha}_2=\vec{\alpha}\Rightarrow\vec{\alpha}\cdot\vec{\beta}=\vec{\alpha}_1\cdot\vec{\beta}+\vec{\alpha}_2\cdot\vec{\beta}\overset{\vec{\alpha}_2\perp\vec{\beta}}{\Rightarrow}\vec{\alpha}\cdot\vec{\beta}=\vec{\alpha}_1\cdot\vec{\beta}$

Διακρίνουμε τις περιπτώσεις:

α) $\vec{\alpha}_1\uparrow\uparrow\vec{\beta}\Rightarrow\text{συν}\left(\widehat{\vec{\alpha}_1,\vec{\beta}}\right)=1$

Οπότε $\vec{\alpha}_1\cdot\vec{\beta}=\vec{\alpha}\cdot\vec{\beta}\Rightarrow\left|\vec{\alpha}_1\right|\cdot\left|\vec{\beta}\right|=\vec{\alpha}\cdot\vec{\beta}\Rightarrow\left|\vec{\alpha}_1\right|=\frac{\vec{\alpha}\cdot\vec{\beta}}{\left|\vec{\beta}\right|}\qquad(1)$

όμως ισχύει $\vec{\alpha}_1\left|\vec{\beta}\right|=\left|\vec{\alpha}_1\right|\cdot\vec{\beta}$ άρα $\vec{\alpha}_1=\frac{\left|\vec{\alpha}_1\right|\cdot\vec{\beta}}{\left|\vec{\beta}\right|}\overset{(1)}{=}\frac{\left(\vec{\alpha}\cdot\vec{\beta}\right)}{\left|\vec{\beta}\right|^2}$

και αφού $\vec{\alpha}_1+\vec{\alpha}_2=\vec{\alpha}$ είναι $\vec{\alpha}_2=\vec{\alpha}-\frac{\left(\vec{\alpha}\cdot\vec{\beta}\right)\cdot\vec{\beta}}{\left|\vec{\beta}\right|^2}$

β) $\vec{\alpha}_1 \uparrow\downarrow \vec{\beta} \Rightarrow συν\left(\widehat{\vec{\alpha}_1, \vec{\beta}}\right) = -1 \Rightarrow \vec{\alpha}_1 \cdot \vec{\beta} = -|\vec{\alpha}_1| \cdot |\vec{\beta}|$

Οπότε $\vec{\alpha}_1 \cdot \vec{\beta} = \vec{\alpha} \cdot \vec{\beta} \Rightarrow -|\vec{\alpha}_1| \cdot |\vec{\beta}| = \vec{\alpha} \cdot \vec{\beta} \Rightarrow |\vec{\alpha}_1| = -\dfrac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\beta}|} \qquad (2)$

Όμως ισχύει $\vec{\alpha}_1 \cdot |\vec{\beta}| = -|\vec{\alpha}_1| \cdot \vec{\beta}$ άρα $\vec{\alpha}_1 = -\dfrac{|\vec{\alpha}_1| \cdot \vec{\beta}}{|\vec{\beta}|} \overset{(2)}{=} \dfrac{(\vec{\alpha} \cdot \vec{\beta}) \cdot \vec{\beta}}{|\vec{\beta}|^2}$

και αφού $\vec{\alpha}_1 + \vec{\alpha}_2 = \vec{\alpha}$ είναι $\vec{\alpha}_2 = \vec{\alpha} - \dfrac{(\vec{\alpha} \cdot \vec{\beta}) \cdot \vec{\beta}}{|\vec{\beta}|^2}$

6.Δ8

Το ΑΒΓΔΕΖ είναι κανονικό εξάγωνο άρα ισχύει :

$\overrightarrow{ΒΓ} = \overrightarrow{ΖΕ} \Rightarrow \overrightarrow{ΟΓ} - \overrightarrow{ΟΒ} = \overrightarrow{ΟΕ} - \overrightarrow{ΟΖ} \Rightarrow \vec{\gamma} - \vec{\beta} = \overrightarrow{ΟΕ} - \overrightarrow{ΟΖ} \quad (1)$

$\overrightarrow{ΑΒ} = \overrightarrow{ΕΔ} \Rightarrow \overrightarrow{ΟΒ} - \overrightarrow{ΟΑ} = \overrightarrow{ΟΔ} - \overrightarrow{ΟΕ} \Rightarrow \vec{\beta} - \vec{\alpha} = \overrightarrow{ΟΔ} - \overrightarrow{ΟΕ} \quad (2)$

$\overrightarrow{ΑΖ} = \overrightarrow{ΓΔ} \Rightarrow \overrightarrow{ΟΖ} - \overrightarrow{ΟΑ} = \overrightarrow{ΟΔ} - \overrightarrow{ΟΓ} \Rightarrow \overrightarrow{ΟΖ} - \vec{\alpha} = \overrightarrow{ΟΔ} - \vec{\gamma} \quad (3)$

Λύνοντας τις (1) , (2) , (3) ως σύστημα με αγνώστους τα $\overrightarrow{ΟΔ}, \overrightarrow{ΟΕ}, \overrightarrow{ΟΖ}$ βρίσκουμε

$\overrightarrow{ΟΔ} = \vec{\alpha} - 2\vec{\beta} + 2\vec{\gamma}$, $\overrightarrow{ΟΕ} = 2\vec{\alpha} - 3\vec{\beta} + 2\vec{\gamma}$, $\overrightarrow{ΟΖ} = 2\vec{\alpha} - 2\vec{\beta} + \vec{\gamma}$