Γ ΛΥΚΕΙΟΥ ΜΕΡΟΣ Β

31.63

Έστω $I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{x}{\eta\mu^2x\,\sigma\upsilon\nu^2x}dx$. Τότε

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{x}{\eta\mu^2x\,\sigma\upsilon\nu^2x}dx\overset{\substack{\eta\mu(\pi-x)=\eta\mu x\\ \sigma\upsilon\nu(\pi-x)=-\sigma\upsilon\nu x}}{=}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{x}{\eta\mu^2(\pi-x)\sigma\upsilon\nu^2(\pi-x)}dx=$$

$$\overset{\substack{\text{θέτουμε } y=\pi-x\Rightarrow x=\pi-y\\ \Rightarrow dy=-dx\\ \text{για } x=\frac{\pi}{6}\Rightarrow y=\pi-\frac{\pi}{6}=\frac{\pi}{3}\\ \text{για } x=\frac{\pi}{3}\Rightarrow y=\pi-\frac{\pi}{3}=\frac{\pi}{6}}}{=}\quad -\int_{\frac{\pi}{3}}^{\frac{\pi}{6}}\frac{\pi-y}{\eta\mu^2y\,\sigma\upsilon\nu^2y}dy=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\pi-y}{\eta\mu^2y\,\sigma\upsilon\nu^2y}dy=$$

$$=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2y\,\sigma\upsilon\nu^2y}dy-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{y}{\eta\mu^2y\,\sigma\upsilon\nu^2y}dy=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2x\,\sigma\upsilon\nu^2x}dx-I$$

Επομένως

$$I=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2x\,\sigma\upsilon\nu^2x}dx-I\Rightarrow 2I=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2x\,\sigma\upsilon\nu^2x}dx\Rightarrow I=\frac{\pi}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2x\,\sigma\upsilon\nu^2x}dx \quad (1)$$

Οπότε τώρα πρέπει να υπολογίσουμε το ολοκλήρωμα $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2x\,\sigma\upsilon\nu^2x}$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2x\sigma\upsilon\nu^2x}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2x}\cdot\frac{1}{\sigma\upsilon\nu^2x}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2x}(\varepsilon\varphi x)'\,dx=$$

$$=\left[\frac{1}{\eta\mu^2x}\varepsilon\varphi x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{1}{\eta\mu^2x}\right)'\varepsilon\varphi x dx=$$

$$=\frac{1}{\eta\mu^2\frac{\pi}{3}}\varepsilon\varphi\frac{\pi}{3}-\frac{1}{\eta\mu^2\frac{\pi}{6}}\varepsilon\varphi\frac{\pi}{6}-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\frac{2\,\cancel{\eta\mu x}\,\sigma\upsilon\nu x}{\eta\mu^{\cancel{4}3}x}\varepsilon\varphi x dx=$$

$$=\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}\sqrt{3}-\frac{1}{\left(\frac{1}{2}\right)^2}\frac{\sqrt{3}}{3}-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\frac{2\,\cancel{\sigma\upsilon\nu x}}{\eta\mu^{\cancel{3}2}x}\,\frac{\cancel{\eta\mu x}}{\cancel{\sigma\upsilon\nu x}}dx=$$

$$=\frac{1}{\frac{3}{4}}\sqrt{3}-\frac{1}{\frac{1}{4}}\frac{\sqrt{3}}{3}-\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}-\frac{2}{\eta\mu^2x}dx=\frac{4\cancel{\sqrt{3}}}{\cancel{3}}-\frac{4\cancel{\sqrt{3}}}{\cancel{3}}+2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2x}dx=$$

$$=2\left[-\sigma\varphi x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=2\left(-\sigma\varphi\frac{\pi}{3}+\sigma\varphi\frac{\pi}{6}\right)=2\left(-\frac{\sqrt{3}}{3}+\sqrt{3}\right)=\frac{4\sqrt{3}}{3}$$

Άρα $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\eta\mu^2 x\,\sigma\upsilon\nu^2 x}dx=\frac{4\sqrt{3}}{3}$ (2)

Οπότε

$$(1),(2)\Rightarrow I=\frac{\pi}{\cancel{2}}\,\frac{\overset{2}{\cancel{4}}\sqrt{3}}{3}\Rightarrow I=\frac{2\pi\sqrt{3}}{3}$$