31.58

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\eta\mu x \sigma\upsilon\nu x}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sigma\upsilon\nu x}{\eta\mu x \sigma\upsilon\nu^2 x}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sigma\upsilon\nu x}{\eta\mu x\left(1-\eta\mu^2 x\right)}dx =$$

θέτουμε $y=\eta\mu x \Rightarrow dy=\sigma\upsilon\nu dx$

για $x=\frac{\pi}{4} \Rightarrow y=\eta\mu\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

για $x=\frac{\pi}{3} \Rightarrow y=\eta\mu\frac{\pi}{3}=\frac{1}{2}$

$$= \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{y\left(1-y^2\right)}dy = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{y(1-y)(y+1)}dy$$

Οπότε θα πρέπει να βρούμε τα Α , Β και Γ ώστε

$$\frac{1}{y(1-y)(y+1)} = \frac{A}{y}+\frac{B}{1-y}+\frac{\Gamma}{y+1}$$

Έχουμε

$$\frac{1}{y(1-y)(y+1)} = \frac{A}{y}+\frac{B}{1-y}+\frac{\Gamma}{y+1} \Rightarrow$$

$$\Rightarrow \cancel{y(1-y)(y+1)}\frac{1}{\cancel{y(1-y)(y+1)}} = \cancel{y}(1-y)(y+1)\frac{A}{\cancel{y}}+y\cancel{(1-y)}(y+1)\frac{B}{\cancel{1-y}}+$$

$$+y(1-y)\cancel{(y+1)}\frac{\Gamma}{\cancel{y+1}} \Rightarrow$$

$$\Rightarrow 1 = A(1-y)(y+1)+By(y+1)+\Gamma y(1-y) \Rightarrow$$

$$\Rightarrow 1 = A\left(1-y^2\right)+B\left(y^2+y\right)+\Gamma\left(y-y^2\right) \Rightarrow$$

$$\Rightarrow 1 = A-Ay^2+By^2+By+\Gamma y-\Gamma y^2 \Rightarrow$$

$$\Rightarrow 1 = (-A+B-\Gamma)y^2+(B+\Gamma)y+A \Rightarrow$$

$$\Rightarrow \begin{cases} -A+B-\Gamma=0 \\ B+\Gamma=0 \\ A=1 \end{cases} \overset{\text{λύνουμε το σύστημα}}{\Rightarrow} \begin{cases} A=1 \\ B=\frac{1}{2} \\ \Gamma=-\frac{1}{2} \end{cases}$$

Οπότε $\frac{1}{y(1-y)(y+1)} = \frac{1}{y}+\frac{\frac{1}{2}}{1-y}-\frac{\frac{1}{2}}{y+1}$

Άρα

$$\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{y(1-y)(y+1)}dy = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{y}+\frac{\frac{1}{2}}{1-y}-\frac{\frac{1}{2}}{y+1}dy =$$

$$=\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{y}dy+\frac{1}{2}\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{1-y}dy-\frac{1}{2}\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{y+1}dy=$$

$$=\left[\ln|y|\right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}-\frac{1}{2}\left[\ln|1-y|\right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}-\frac{1}{2}\left[\ln|y+1|\right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}=$$

$$=\ln\left|\frac{\sqrt{3}}{2}\right|-\ln\left|\frac{\sqrt{2}}{2}\right|-\frac{1}{2}\left(\ln\left|1-\frac{\sqrt{3}}{2}\right|-\ln\left|1-\frac{\sqrt{2}}{2}\right|\right)-\frac{1}{2}\left(\ln\left|\frac{\sqrt{3}}{2}+1\right|-\ln\left|\frac{\sqrt{2}}{2}+1\right|\right)=$$

$$=\ln\frac{\frac{\sqrt{3}}{\cancel{2}}}{\frac{\sqrt{2}}{\cancel{2}}}-\frac{1}{2}\left(\ln\frac{2-\sqrt{3}}{2}-\ln\frac{2-\sqrt{2}}{2}\right)-\frac{1}{2}\left(\ln\frac{2+\sqrt{3}}{2}-\ln\frac{2+\sqrt{2}}{2}\right)=$$

$$=\ln\frac{\sqrt{3}}{\sqrt{2}}-\frac{1}{2}\left(\ln\frac{\frac{2-\sqrt{3}}{\cancel{2}}}{\frac{2-\sqrt{2}}{\cancel{2}}}\right)-\frac{1}{2}\left(\ln\frac{\frac{2+\sqrt{3}}{\cancel{2}}}{\frac{2+\sqrt{2}}{\cancel{2}}}\right)=$$

$$=\ln\left(\frac{3}{2}\right)^{\frac{1}{2}}-\frac{1}{2}\left(\ln\frac{2-\sqrt{3}}{2-\sqrt{2}}\right)-\frac{1}{2}\left(\ln\frac{2+\sqrt{3}}{2+\sqrt{2}}\right)=$$

$$=\frac{1}{2}\ln\frac{3}{2}-\frac{1}{2}\left(\ln\frac{2-\sqrt{3}}{2-\sqrt{2}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{2}}\right)=\frac{1}{2}\ln\frac{3}{2}-\frac{1}{2}\left(\ln\frac{2^2-\sqrt{3}^2}{2^2-\sqrt{2}^2}\right)=$$

$$=\frac{1}{2}\ln\frac{3}{2}-\frac{1}{2}\left(\ln\frac{1}{2}\right)=\frac{1}{2}\ln\frac{\frac{3}{\cancel{2}}}{\frac{1}{\cancel{2}}}=\frac{1}{2}\ln 3=\ln\sqrt{3}$$