Γ ΛΥΚΕΙΟΥ ΜΕΡΟΣ Β

31.26 1)

$$\int_0^{\ln 2}\frac{e^x-1}{e^x+2}dx \overset{\substack{\text{θέτουμε } y=e^x\Rightarrow dy=e^x dx\Rightarrow dy=ydx\Rightarrow \boxed{dx=\frac{dy}{y}}\\ \text{για } x=0\Rightarrow y=e^0=1\\ \text{για } x=\ln 2\Rightarrow y=e^{\ln 2}=2}}{=} \int_1^2\frac{y-1}{y+2}\frac{1}{y}dy=\int_1^2\frac{y-1}{y(y+2)}dy \quad (1)$$

Θα πρέπει να βρούμε τα Α και Β ώστε $\frac{y-1}{y(y+2)}=\frac{A}{y}+\frac{B}{y+2}$

Έχουμε

$$\frac{y-1}{y(y+2)}=\frac{A}{y}+\frac{B}{y+2}\Rightarrow$$

$$\Rightarrow \cancel{y(y+2)}\frac{y-1}{\cancel{y(y+2)}}=\cancel{y}(y+2)\frac{A}{\cancel{y}}+y\cancel{(y+2)}\frac{B}{\cancel{y+2}}\Rightarrow$$

$$\Rightarrow y-1=A(y+2)+By\Rightarrow y-1=Ay+2A+By\Rightarrow$$

$$\Rightarrow y-1=(A+B)y+2A\Rightarrow\begin{cases}A+B=1\\2A=-1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=-\frac{1}{2}\\B=\frac{3}{2}\end{cases}$$

Οπότε $\frac{y-1}{y(y+2)}=\frac{-\frac{1}{2}}{y}+\frac{\frac{3}{2}}{y+2}$ (2)

Άρα

$$(1),(2)\Rightarrow\int_1^2\frac{y-1}{y(y+2)}dy=\int_1^2\frac{-\frac{1}{2}}{y}+\frac{\frac{3}{2}}{y+2}dy=-\frac{1}{2}\int_1^2\frac{1}{y}dy+\frac{3}{2}\int_1^2\frac{1}{y+2}dy=$$

$$=-\frac{1}{2}\left[\ln|y|\right]_1^2+\frac{3}{2}\left[\ln|y+2|\right]_1^2=-\frac{1}{2}\left(\ln|2|-\cancel{\ln|1|}\right)+\frac{3}{2}\left(\ln|2+2|-\ln|1+2|\right)=$$

$$=-\frac{1}{2}\ln 2+\frac{3}{2}\ln 4-\frac{3}{2}\ln 3=\frac{-\ln 2+6\ln 2-3\ln 3}{2}=\frac{5\ln 2-3\ln 3}{2}$$

31.26 2)

Έστω $I=\int_0^{\ln 2}\frac{e^x-2}{e^x+1}dx$. Έχουμε

$$I=\int_0^{\ln 2}\frac{e^x-2}{e^x+1}dx \overset{\substack{\text{θέτουμε } y=e^x\Rightarrow dy=e^x dx\Rightarrow dy=ydx\Rightarrow \boxed{dx=\frac{dy}{y}}\\ \text{για } x=0\Rightarrow y=e^0=1\\ \text{για } x=\ln 2\Rightarrow y=e^{\ln 2}=2}}{=} \int_1^2\frac{y-2}{y+1}\frac{1}{y}dy\Rightarrow I=\int_1^2\frac{y-2}{y(y+1)}dy \quad (1)$$

Θα πρέπει να βρούμε τα Α και Β ώστε $\frac{y-2}{y(y+1)}=\frac{A}{y}+\frac{B}{y+1}$

Έχουμε

$$\frac{y-2}{y(y+1)}=\frac{A}{y}+\frac{B}{y+1}\Rightarrow$$

$$\Rightarrow \cancel{y(y+1)}\frac{y-2}{\cancel{y(y+1)}}=\cancel{y}(y+1)\frac{A}{\cancel{y}}+y\cancel{(y+1)}\frac{B}{\cancel{y+1}}\Rightarrow$$

$$\Rightarrow y-2=A(y+1)+By\Rightarrow y-2=Ay+A+By\Rightarrow$$

$$\Rightarrow y-2=(A+B)y+A\Rightarrow\begin{cases}A+B=1\\A=-2\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=-2\\B=3\end{cases}$$

Οπότε $\frac{y-2}{y(y+1)}=\frac{-2}{y}+\frac{3}{y+1}$ (2)

Άρα

$$(1),(2)\Rightarrow\int_1^2\frac{y-2}{y(y+1)}dy=\int_1^2\frac{-2}{y}+\frac{3}{y+1}dy=-2\int_1^2\frac{1}{y}dy+3\int_1^2\frac{1}{y+1}dy=$$

$$=-2\left[\ln|y|\right]_1^2+3\left[\ln|y+1|\right]_1^2=-2\left(\ln|2|-\cancel{\ln|1|}\right)+3\left(\ln|2+1|-\ln|1+1|\right)=$$

$$=-2\ln2+3\ln3-3\ln2=3\ln3-5\ln2$$

31.26 3)

Έστω $I=\int_0^{\ln2}\frac{dx}{e^x+1}$. Έχουμε

$$I=\int_0^{\ln2}\frac{dx}{e^x+1}\overset{\substack{\text{θέτουμε } y=e^x\Rightarrow dy=e^x dx\Rightarrow dy=ydx\Rightarrow \boxed{dx=\frac{dy}{y}}\\ \text{για } x=0\Rightarrow y=e^0=1\\ \text{για } x=\ln2\Rightarrow y=e^{\ln2}=2}}{=}\int_1^2\frac{1}{y+1}\frac{1}{y}dy\Rightarrow I=\int_1^2\frac{1}{y(y+1)}dy\ (1)$$

Θα πρέπει να βρούμε τα Α και Β ώστε $\frac{1}{y(y+1)}=\frac{A}{y}+\frac{B}{y+1}$

Έχουμε

$$\frac{1}{y(y+1)}=\frac{A}{y}+\frac{B}{y+1}\Rightarrow$$

$$\Rightarrow \cancel{y(y+1)}\frac{1}{\cancel{y(y+1)}}=\cancel{y}(y+1)\frac{A}{\cancel{y}}+y\cancel{(y+1)}\frac{B}{\cancel{y+1}}\Rightarrow$$

$$\Rightarrow1=A(y+1)+By\Rightarrow1=Ay+A+By\Rightarrow$$

$$\Rightarrow 1=(A+B)y+A\Rightarrow\begin{cases}A+B=0\\A=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=1\\B=-1\end{cases}$$

Οπότε $\dfrac{1}{y(y+1)}=\dfrac{1}{y}-\dfrac{1}{y+1}$ (2)

Άρα

$$(1),(2)\Rightarrow I=\int_1^2\frac{1}{y(y+1)}dy=\int_1^2\frac{1}{y}-\frac{1}{y+1}dy=\int_1^2\frac{1}{y}dy-\int_1^2\frac{1}{y+1}dy=$$

$$=\left[\ln|y|\right]_1^2-\left[\ln|y+1|\right]_1^2=\left(\ln|2|-\cancel{\ln|1|}\right)-\left(\ln|2+1|-\ln|1+1|\right)=$$

$$=\ln 2-\ln 3+\ln 2=2\ln 2-\ln 3=\ln\frac{4}{3}$$

31.26 4)

Έστω $I=\int_0^{\ln 5}\frac{1}{e^x+3}dx$. Έχουμε

$$I=\int_0^{\ln 5}\frac{1}{e^x+3}dx\overset{\substack{\text{θέτουμε } y=e^x\Rightarrow dy=e^x dx\Rightarrow dy=ydx\Rightarrow \boxed{dx=\frac{dy}{y}}\\ \text{για } x=0\Rightarrow y=e^0=1\\ \text{για } x=\ln 5\Rightarrow y=e^{\ln 5}=5}}{=}\int_1^5\frac{1}{y+3}\frac{1}{y}dy\Rightarrow I=\int_1^5\frac{1}{y(y+3)}dy\ (1)$$

Θα πρέπει να βρούμε τα Α και Β ώστε $\dfrac{1}{y(y+3)}=\dfrac{A}{y}+\dfrac{B}{y+3}$

Έχουμε

$$\frac{1}{y(y+3)}=\frac{A}{y}+\frac{B}{y+3}\Rightarrow$$

$$\Rightarrow \cancel{y(y+3)}\frac{1}{\cancel{y(y+3)}}=\cancel{y}(y+3)\frac{A}{\cancel{y}}+y\cancel{(y+3)}\frac{B}{\cancel{y+3}}\Rightarrow$$

$$\Rightarrow 1=A(y+3)+By\Rightarrow 1=Ay+3A+By\Rightarrow$$

$$\Rightarrow 1=(A+B)y+3A\Rightarrow\begin{cases}A+B=0\\3A=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=\frac{1}{3}\\B=-\frac{1}{3}\end{cases}$$

Οπότε $\dfrac{1}{y(y+3)}=\dfrac{\frac{1}{3}}{y}-\dfrac{\frac{1}{3}}{y+3}$ (2)

Άρα

$$(1),(2)\Rightarrow I=\int_1^5\frac{1}{y(y+3)}dy=\int_1^5\frac{\frac{1}{3}}{y}-\frac{\frac{1}{3}}{y+3}dy=\frac{1}{3}\int_1^5\frac{1}{y}dy-\frac{1}{3}\int_1^5\frac{1}{y+3}dy=$$

$$=\frac{1}{3}\left[\ln|y|\right]_1^5-\frac{1}{3}\left[\ln|y+3|\right]_1^5=\frac{1}{3}\left(\ln|5|-\cancel{\ln|1|}\right)-\frac{1}{3}\left(\ln|5+3|-\ln|1+3|\right)=$$

$$=\frac{\ln 5}{3}-\frac{\ln 8-\ln 4}{3}=\frac{\ln 5}{3}-\frac{3\ln 2-2\ln 2}{3}=\frac{\ln 5-\ln 2}{3}$$

31.26 5)

$$\int_0^{\ln 2}\frac{e^{2x}}{1+e^x}dx \overset{\substack{\text{θέτουμε } y=e^x\Rightarrow dy=e^x dx\Rightarrow dy=ydx\Rightarrow \boxed{dx=\frac{dy}{y}}\\ \text{για } x=0\Rightarrow y=e^0=1\\ \text{για } x=\ln 2\Rightarrow y=e^{\ln 2}=2}}{=} \int_1^2\frac{y^{\cancel{2}}}{1+y}\frac{1}{\cancel{y}}dy=\int_1^2\frac{y}{y+1}dy=$$

$$=\int_1^2\frac{y+1-1}{y+1}dy=\int_1^2\frac{y+1}{y+1}-\frac{1}{y+1}dy=\int_1^2\left(1-\frac{1}{y+1}\right)dy=\left[y-\ln|y+1|\right]_1^2=$$

$$=2-\ln|2+1|-1+\ln|1+1|=1-\ln 3+\ln 2=1+\ln\frac{2}{3}$$

31.26 6)

$$\int_0^1\frac{e^{3x}-2e^{2x}+e^x}{e^x-1}dx \overset{\substack{\text{θέτουμε } y=e^x\Rightarrow dy=e^x dx\Rightarrow dy=ydx\Rightarrow \boxed{dx=\frac{dy}{y}}\\ \text{για } x=0\Rightarrow y=e^0=1\\ \text{για } x=1\Rightarrow y=e^1=e}}{=} \int_1^e\frac{y^3-2y^2+y}{y-1}\frac{1}{y}dy=$$

$$=\int_1^e\frac{\cancel{y}\left(y^2-2y+1\right)}{y-1}\frac{1}{\cancel{y}}dy=\int_0^e\frac{(y-1)^{\cancel{2}}}{\cancel{y-1}}dy=\int_1^e(y-1)dy=\left[\frac{y^2}{2}-y\right]_1^e=$$

$$=\frac{e^2}{2}-e-\frac{1^2}{2}+1=\frac{e^2-2e+1}{2}$$

31.26 7)

Έστω $I=\int_0^{2\ln 2}\frac{e^{2x}-1}{e^x+2}dx$. Έχουμε

$$I=\int_0^{2\ln 2}\frac{e^{2x}-1}{e^x+2}dx \overset{\substack{\text{θέτουμε } y=e^x\Rightarrow dy=e^x dx\Rightarrow dy=ydx\Rightarrow \boxed{dx=\frac{dy}{y}}\\ \text{για } x=0\Rightarrow y=e^0=1\\ \text{για } x=2\ln 2\Rightarrow y=e^{2\ln 2}=4}}{=} \int_1^4\frac{y^2-1}{y+2}\frac{1}{y}dy=\int_1^4\frac{y^2-1}{y^2+2y}dy=$$

$$=\int_1^4\frac{y^2+2y-2y-1}{y^2+2y}dy=\int_1^4\frac{y^2+2y}{y^2+2y}+\frac{-2y-1}{y^2+2y}dy=\int_1^4 1+\frac{-2y-1}{y^2+2y}dy=$$

$$=\int_1^4 1dx+\int_1^4\frac{-2y-1}{y^2+2y}dy=1\cdot(4-1)+\int_1^4\frac{-2y-1}{y(y+2)}dy=3-\int_1^4\frac{2y+1}{y(y+2)}dy$$

Οπότε $I=3-\int_1^4\frac{2y+1}{y(y+2)}dy$ (1)

Θα πρέπει να βρούμε τα A και B ώστε $\frac{2y+1}{y(y+2)}=\frac{A}{y}+\frac{B}{y+2}$

Έχουμε

$$\frac{2y+1}{y(y+2)}=\frac{A}{y}+\frac{B}{y+2}\Rightarrow$$

$$\Rightarrow \cancel{y(y+2)}\frac{2y+1}{\cancel{y(y+2)}}=\cancel{y}(y+2)\frac{A}{\cancel{y}}+y\cancel{(y+2)}\frac{B}{\cancel{y+2}}\Rightarrow$$

$$\Rightarrow 2y+1=A(y+2)+By\Rightarrow 2y+1=Ay+2A+By\Rightarrow$$

$$\Rightarrow 2y+1=(A+B)y+2A\Rightarrow\begin{cases}A+B=2\\2A=1\end{cases}\xRightarrow{\text{λύνουμε το σύστημα}}\begin{cases}A=\frac{1}{2}\\B=\frac{3}{2}\end{cases}$$

Οπότε $\frac{2y+1}{y(y+2)}=\frac{\frac{1}{2}}{y}+\frac{\frac{3}{2}}{y+2}$ (2)

Άρα

$$(1),(2)\Rightarrow I=3-\int_1^4\frac{\frac{1}{2}}{y}+\frac{\frac{3}{2}}{y+2}dy=3-\frac{1}{2}\int_1^4\frac{1}{y}dy-\frac{3}{2}\int_1^4\frac{1}{y+2}dy=$$

$$=3-\frac{1}{2}\left[\ln|y|\right]_1^4-\frac{3}{2}\left[\ln|y+2|\right]_1^4=$$

$$=3-\frac{1}{2}\left(\ln|4|-\cancel{\ln|1|}\right)-\frac{3}{2}\left(\ln|4+2|-\ln|1+2|\right)=$$

$$=3-\frac{1}{2}\ln 4-\frac{3}{2}(\ln 6-\ln 3)=3-\frac{2\ln 2}{2}-\frac{3\ln 2}{2}=3-\frac{5\ln 2}{2}$$

31.26 8)

Έστω $I=\int_0^{\ln 2}\frac{1}{e^{2x}+3e^x+2}dx$. Έχουμε

$$I=\int_0^{\ln 2}\frac{1}{e^{2x}+3e^x+2}dx\ \overset{\substack{\text{θέτουμε } y=e^x\Rightarrow dy=e^x dx\Rightarrow dy=ydx\Rightarrow \boxed{dx=\frac{dy}{y}}\\ \text{για } x=0\Rightarrow y=e^0=1\\ \text{για } x=\ln 2\Rightarrow y=e^{\ln 2}=2}}{=}\ \int_1^2\frac{1}{y^2+3y+2}\frac{1}{y}dy=$$

$$=\int_1^2\frac{1}{y(y^2+3y+2)}dy\ \overset{\substack{\Delta=1,\ y_{1,2}=\frac{-3\pm1}{2}=\ \nearrow y_1=\frac{-3-1}{2}\Rightarrow y_1=-2\\ \searrow y_2=\frac{-3+1}{2}\Rightarrow y_2=-1}}{=}\ \int_1^2\frac{1}{y(y+1)(y+2)}dy$$

Οπότε $I=\int_1^2\frac{1}{y(y+1)(y+2)}dy$ (1)

Θα πρέπει να βρούμε τα Α , Β και Γ ώστε

$$\frac{1}{y(y+1)(y+2)}=\frac{A}{y}+\frac{B}{y+1}+\frac{\Gamma}{y+2}$$

Έχουμε

$$\frac{1}{y(y+1)(y+2)}=\frac{A}{y}+\frac{B}{y+1}+\frac{\Gamma}{y+2}\Rightarrow$$

$$\Rightarrow \cancel{y(y+1)(y+2)}\,\frac{1}{\cancel{y(y+1)(y+2)}}=\cancel{y}\,(y+1)(y+2)\frac{A}{\cancel{y}}+y\cancel{(y+1)}\,(y+2)\frac{B}{\cancel{y+1}}+$$

$$+y(y+1)\,\cancel{(y+2)}\,\frac{\Gamma}{\cancel{y+2}}\Rightarrow$$

$$\Rightarrow 1=A(y+1)(y+2)+By(y+2)+\Gamma y(y+1)\Rightarrow$$

$$\Rightarrow 1=A\left(y^2+3y+2\right)+B\left(y^2+2y\right)+\Gamma\left(y^2+y\right)\Rightarrow$$

$$\Rightarrow 1=Ay^2+3Ay+2A+By^2+2By+\Gamma y^2+\Gamma y\Rightarrow$$

$$\Rightarrow 1=(A+B+\Gamma)y^2+(3A+2B+\Gamma)y+2A\Rightarrow$$

$$\Rightarrow\begin{cases}A+B+\Gamma=0\\3A+2B+\Gamma=0\\2A=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=\dfrac{1}{2}\\B=-1\\\Gamma=\dfrac{1}{2}\end{cases}$$

Οπότε $\dfrac{1}{y(y+1)(y+2)}=\dfrac{\frac{1}{2}}{y}-\dfrac{1}{y+1}+\dfrac{\frac{1}{2}}{y+2}$ (2)

Άρα

$$(1),(2)\Rightarrow I=\int_1^2\frac{\frac{1}{2}}{y}-\frac{1}{y+1}+\frac{\frac{1}{2}}{y+2}dy=\frac{1}{2}\int_1^2\frac{1}{y}dy-\int_1^2\frac{1}{y+1}dy+\frac{1}{2}\int_1^2\frac{1}{y+2}dy=$$

$$=\frac{1}{2}\left[\ln|y|\right]_1^2-\left[\ln|y+1|\right]_1^2+\frac{1}{2}\left[\ln|y+2|\right]_1^2=$$

$$=\frac{1}{2}\left(\ln|2|-\cancel{\ln|1|}\right)-\ln|2+1|+\ln|1+1|+\frac{1}{2}\left(\ln|2+2|-\ln|1+2|\right)=$$

$$=\frac{1}{2}\ln 2-\ln 3+\ln 2+\frac{1}{2}\ln 4-\frac{1}{2}\ln 3=\frac{5\ln 2-3\ln 3}{2}$$