Γ ΛΥΚΕΙΟΥ ΜΕΡΟΣ Β

31.25 1)

α) $\int_1^e x^2 \ln x dx = \int_1^e \left(\frac{x^3}{3}\right)' \ln x dx = \left[\frac{x^3 \ln x}{3}\right]_1^e - \int_1^e \frac{x^3}{3}(\ln x)' dx =$

$$= \frac{e^3 \ln e}{3} - \int_1^e \frac{x^{\cancel{3}2}}{3}\frac{1}{\cancel{x}}dx = \frac{e^3}{3} - \frac{1}{3}\int_1^e x^2 dx = \frac{e^3}{3} - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^e = \frac{e^3}{3} - \frac{e^3-1}{9} = \frac{2e^3+1}{9}$$

β) $\int_1^2 \ln x dx = \int_1^2 (x)' \ln x dx = [x \ln x]_1^2 - \int_1^2 x(\ln x)' dx = 2\ln 2 - \int_1^2 \cancel{x}\frac{1}{\cancel{x}}dx = 2\ln 2 - 1$

γ) Μια αρχική της $\frac{1}{\sqrt{x}}$ είναι η $2\sqrt{x}$. Έτσι

$$\int_1^4 \frac{\ln x}{\sqrt{x}}dx = \int_1^4 \left(2\sqrt{x}\right)' \ln x dx = \left[2\sqrt{x}\ln x\right]_1^4 - \int_1^4 2\sqrt{x}(\ln x)' dx =$$

$$= 2\sqrt{4}\ln 4 - \int_1^4 2\sqrt{x}\frac{1}{x}dx = 4\ln 4 - 2\int_1^4 \frac{1}{\sqrt{x}}dx = 4\ln 4 - 2\left[2\sqrt{x}\right]_1^4 = 4\ln 4 - 6$$

31.25 2)

$$\int_1^2 x^2 \ln x dx = \int_1^2 \left(\frac{x^3}{3}\right)' \ln x dx = \left[\frac{x^3 \ln x}{3}\right]_1^2 - \int_1^2 \frac{x^3}{3}(\ln x)' dx =$$

$$= \frac{2^3 \ln 2}{3} - \cancel{\frac{1^3 \ln 1}{3}} - \int_1^2 \frac{x^{\cancel{3}2}}{3}\frac{1}{\cancel{x}}dx = \frac{8\ln 2}{3} - \frac{1}{3}\int_1^2 x^2 dx = \frac{8\ln 2}{3} - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^2 =$$

$$= \frac{8\ln 2}{3} - \frac{1}{3}\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{24\ln 2 - 7}{9}$$

31.25 3)

$$\int_1^e \ln x dx = \int_1^e (x)' \ln x dx = [x \ln x]_1^e - \int_1^e x(\ln x)' dx = e - \int_1^e \cancel{x}\frac{1}{\cancel{x}}dx =$$

$$= e - 1(e-1) = 1$$

31.25 4)

$$\int_0^1 x\ln(x+1)dx = \int_0^1 \left(\frac{x^2}{2}\right)' \ln(x+1)dx = \left[\frac{x^2 \ln(x+1)}{2}\right]_0^1 - \int_0^1 \frac{x^2}{2}\left[\ln(x+1)\right]' dx =$$

$$= \frac{\ln 2}{2} - \frac{1}{2}\int_0^1 \frac{x^2}{x+1}dx = \frac{\ln 2}{2} - \frac{1}{2}\int_0^1 \frac{x^2-1+1}{x+1}dx =$$

$$= \frac{\ln 2}{2} - \frac{1}{2}\int_0^1 \frac{x^2-1}{x+1} + \frac{1}{x+1}dx = \frac{\ln 2}{2} - \frac{1}{2}\int_0^1 \frac{(x-1)\cancel{(x+1)}}{\cancel{x+1}}dx - \frac{1}{2}\int_0^1 \frac{1}{x+1}dx =$$

$$= \frac{\ln 2}{2} - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_0^1 - \frac{1}{2}\left[\ln(x+1)\right]_0^1 = \cancel{\frac{\ln 2}{2}} - \frac{1}{2}\frac{1}{2} - \cancel{\frac{\ln 2}{2}} = \frac{1}{4}$$

31.25 1)

$$\int_0^1 x^2 \ln(x+1)dx = \int_0^1 \left(\frac{x^3}{3}\right)' \ln(x+1)dx = \left[\frac{x^3 \ln(x+1)}{3}\right]_0^1 - \int_0^1 \frac{x^3}{3}\left[\ln(x+1)\right]' dx =$$

$$=\frac{\ln 2}{3}-\frac{1}{3}\int_0^1\frac{x^3}{x+1}dx=\frac{\ln 2}{3}-\frac{1}{3}\int_0^1\frac{x^3+1-1}{x+1}dx=$$

$$=\frac{\ln 2}{3}-\frac{1}{3}\int_0^1\frac{x^3+1}{x+1}-\frac{1}{x+1}dx=\frac{\ln 2}{3}-\frac{1}{3}\int_0^1\frac{\cancel{(x+1)}\left(x^2-x+1\right)}{\cancel{x+1}}dx+\frac{1}{3}\int_0^1\frac{1}{x+1}dx=$$

$$=\frac{\ln 2}{3}-\frac{1}{3}\left[\frac{x^3}{3}-\frac{x^2}{2}+x\right]_0^1+\frac{1}{3}\left[\ln(x+1)\right]_0^1=\frac{\ln 2}{3}-\frac{1}{3}\left(\frac{1}{3}-\frac{1}{2}+1\right)+\frac{\ln 2}{3}=$$

$$=\frac{2\ln 2}{3}-\frac{5}{18}=\frac{12\ln 2-5}{18}$$

31.25 6)

Είναι $\frac{x^2+1}{2x^2}=\frac{\cancel{x^2}}{2\cancel{x^2}}+\frac{1}{2x^2}=\frac{1}{2}+\frac{1}{2x^2}$

μια αρχική της $\frac{1}{2}+\frac{1}{2x^2}$ είναι η $\frac{x}{2}-\frac{1}{2x}$

Οπότε

$$\int_1^2\frac{x^2+1}{2x^2}\ln x dx=\int_1^2\left(\frac{x}{2}-\frac{1}{2x}\right)'\ln x dx=\left[\left(\frac{x}{2}-\frac{1}{2x}\right)\ln x\right]_1^2-\int_1^2\left(\frac{x}{2}-\frac{1}{2x}\right)(\ln x)' dx=$$

$$=\left(1-\frac{1}{4}\right)\ln 2-\int_1^2\left(\frac{x}{2}-\frac{1}{2x}\right)\frac{1}{x}dx=\frac{3}{4}\ln 2-\int_1^2\left(\frac{1}{2}-\frac{1}{2x^2}\right)dx=\frac{3}{4}\ln 2-\left[\frac{x}{2}+\frac{1}{2x}\right]_1^2=$$

$$=\frac{3}{4}\ln 2-\left(\frac{2}{2}+\frac{1}{4}\right)+\left(\frac{1}{2}+\frac{1}{2}\right)=\frac{3}{4}\ln 2-\cancel{1}-\frac{1}{4}+\cancel{1}=\frac{3}{4}\ln 2-\frac{1}{4}$$

31.25 7)

Μια αρχική της $\frac{1}{(x+1)^2}$ είναι η $-\frac{1}{x+1}$

Οπότε αν $I=\int_1^2\frac{\ln(2x-1)}{(x+1)^2}dx$, θα είναι

$$I=\int_1^2\frac{\ln(2x-1)}{(x+1)^2}dx=\int_1^2\left(-\frac{1}{x+1}\right)'\ln(2x-1)dx=$$

$$=\left[\left(-\frac{1}{x+1}\right)\ln(2x-1)\right]_1^2-\int_1^2\left(-\frac{1}{x+1}\right)\left[\ln(2x-1)\right]' dx=$$

$$=\left(-\frac{1}{3}\right)\ln 3-\int_1^2-\frac{1}{x+1}\cdot\frac{2}{2x-1}dx=-\frac{\ln 3}{3}+\int_1^2\frac{2}{(x+1)(2x-1)}dx$$

Οπότε θα πρέπει να βρούμε τα Α και Β ώστε $\frac{2}{(x+1)(2x-1)}=\frac{A}{x+1}+\frac{B}{2x-1}$

Έχουμε

$$\frac{2}{(x+1)(2x-1)}=\frac{A}{x+1}+\frac{B}{2x-1}\Rightarrow$$

$$\Rightarrow \cancel{(x+1)(2x-1)}\frac{2}{\cancel{(x+1)(2x-1)}}=\cancel{(x+1)}(2x-1)\frac{A}{\cancel{x+1}}+(x+1)\cancel{(2x-1)}\frac{B}{\cancel{2x-1}}\Rightarrow$$

$$\Rightarrow 2=A(2x-1)+B(x+1)\Rightarrow 2=2Ax-A+Bx+B\Rightarrow$$

$$\Rightarrow 2=(2A+B)x-A+B\Rightarrow\begin{cases}2A+B=0\\-A+B=2\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=-\frac{2}{3}\\B=\frac{4}{3}\end{cases}$$

Οπότε $\frac{2}{(x+1)(2x-1)}=\frac{-\frac{2}{3}}{x+1}+\frac{\frac{4}{3}}{2x-1}$

Επομένως το ζητούμενο ολοκλήρωμα είναι

$$I=-\frac{\ln 3}{3}+\int_1^2\frac{-\frac{2}{3}}{x+1}+\frac{\frac{4}{3}}{2x-1}dx=-\frac{\ln 3}{3}+\left[-\frac{2}{3}\ln|x+1|+\frac{\cancel{4}^{2}}{3}\frac{\ln|2x-1|}{\cancel{2}}\right]_1^2=$$

$$=-\frac{\ln 3}{3}-\frac{2}{3}\ln|2+1|+\frac{2}{3}\ln|2\cdot 2-1|+\frac{2}{3}\ln|1+1|-\frac{2}{3}\cancel{\ln|2\cdot 1-1|}=$$

$$=-\frac{\ln 3}{3}-\cancel{\frac{2}{3}\ln 3}+\cancel{\frac{2}{3}\ln 3}+\frac{2}{3}\ln 2=\frac{2\ln 2-\ln 3}{3}$$

31.25 8)

Μια αρχική της $\frac{1}{\sqrt{x}}$ είναι η $2\sqrt{x}$

Οπότε

$$\int_1^e\frac{\ln x}{\sqrt{x}}dx=\int_1^e\left(2\sqrt{x}\right)'\ln x dx=\left[2\sqrt{x}\ln x\right]_1^e-\int_1^e 2\sqrt{x}(\ln x)'dx=$$

$$=2\sqrt{e}\ln e-\int_1^e 2\sqrt{x}\frac{1}{x}dx=2\sqrt{e}-2\int_1^e\frac{1}{\sqrt{x}}dx=2\sqrt{e}-2\left[2\sqrt{x}\right]_1^e=$$

$$=2\sqrt{e}-2\left(2\sqrt{e}-2\right)=2\sqrt{e}-4\sqrt{e}+4=4-2\sqrt{e}$$

31.25 9)

Μια αρχική της $\frac{1}{x^3}=x^{-3}$ είναι η $\frac{x^{-2}}{-2}=-\frac{1}{2x^2}$

Οπότε

$$\int_1^e\frac{\ln x}{x^3}dx=\int_1^e\left(-\frac{1}{2x^2}\right)'\ln x dx=\left[-\frac{\ln x}{2x^2}\right]_1^e-\int_1^e-\frac{1}{2x^2}(\ln x)'dx=$$

$$=-\frac{1}{2e^2}+\int_1^e\frac{1}{2x^2}\frac{1}{x}dx=-\frac{1}{2e^2}+\frac{1}{2}\int_1^e\frac{1}{x^3}dx=-\frac{1}{2e^2}+\frac{1}{2}\left[-\frac{1}{2x^2}\right]_1^e=$$

$$=-\frac{1}{2e^2}+\frac{1}{2}\left(-\frac{1}{2e^2}+\frac{1}{2}\right)=-\frac{1}{2e^2}-\frac{1}{4e^2}+\frac{1}{4}=\frac{e^2-3}{4e^2}$$

31.25 10)

Μια αρχική της $\sqrt{x}=x^{\frac{1}{2}}$ είναι η $\frac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2}{3}\sqrt{x^3}$

Οπότε

$$\int_1^e \sqrt{x}\ln x\,dx=\int_1^e\left(\frac{2}{3}\sqrt{x^3}\right)'\ln x dx=\left[\frac{2}{3}\sqrt{x^3}\ln x\right]_1^e-\int_1^e\frac{2}{3}\sqrt{x^3}(\ln x)'\,dx=$$

$$=\frac{2}{3}\sqrt{e^3}\ln e-\int_1^e\frac{2}{3}\sqrt{x^3}\frac{1}{x}dx=\frac{2}{3}\sqrt{e^3}-\frac{2}{3}\int_1^e \cancel{x}\sqrt{x}\frac{1}{\cancel{x}}dx=$$

$$=\frac{2}{3}\sqrt{e^3}-\frac{2}{3}\left[\frac{2}{3}\sqrt{x^3}\right]_1^e=\frac{2}{3}\sqrt{e^3}-\frac{2}{3}\left(\frac{2}{3}\sqrt{e^3}-\frac{2}{3}\right)=\frac{2}{3}\sqrt{e^3}-\frac{4}{9}\sqrt{e^3}+\frac{4}{9}=$$

$$=\frac{2\sqrt{e^3}+4}{3}$$

31.25 11)

$$\int_{-1}^1 \ln\left(\frac{3-x}{3+x}\right)dx=\int_{-1}^1(x)'\ln\left(\frac{3-x}{3+x}\right)dx=$$

$$=\left[x\ln\left(\frac{3-x}{3+x}\right)\right]_{-1}^1-\int_{-1}^1 x\left[\ln\left(\frac{3-x}{3+x}\right)\right]'dx=$$

$$=\ln\left(\frac{3-1}{3+1}\right)+\ln\left(\frac{3+1}{3-1}\right)-\int_{-1}^1 x\frac{\frac{-(3+x)-(3-x)}{(3+x)^{\cancel{2}}}}{\frac{3-x}{\cancel{3+x}}}dx=$$

$$=\ln\frac{1}{2}+\ln 2-\int_{-1}^1\frac{-6x}{(3-x)(3+x)}dx=0-3\int_{-1}^1\frac{-2x}{3-x^2}dx=$$

$$=-3\left[\ln\left|3-x^2\right|\right]_{-1}^1=-3\left(\ln|3-1|-\ln|3-1|\right)=0$$

31.25 12)

$$\int_1^e x\ln^2 x\,dx=\int_1^e\left(\frac{x^2}{2}\right)'\ln^2 x\,dx=\left[\frac{x^2\ln^2 x}{2}\right]_1^e-\int_1^e\frac{x^2}{2}\left(\ln^2 x\right)'dx=$$

$$=\frac{e^2\ln^2 e}{2}-\int_1^e\frac{x^{\cancel{2}}}{\cancel{2}}\cancel{2}\ln x\frac{1}{\cancel{x}}dx=\frac{e^2}{2}-\int_1^e x\ln x\,dx=\frac{e^2}{2}-\int_1^e\left(\frac{x^2}{2}\right)'\ln x\,dx=$$

$$=\frac{e^2}{2}-\left(\left[\frac{x^2\ln x}{2}\right]_1^e-\int_1^e\frac{x^2}{2}(\ln x)'\,dx\right)=\frac{e^2}{2}-\left(\frac{e^2\ln e}{2}-\int_1^e\frac{x^{\cancel{2}}}{2}\frac{1}{\cancel{x}}dx\right)=$$

$$=\frac{e^2}{2}-\left(\frac{e^2}{2}-\frac{1}{2}\int_1^e x\,dx\right)=\frac{\cancel{e^2}}{\cancel{2}}-\frac{\cancel{e^2}}{\cancel{2}}+\frac{1}{2}\int_1^e x\,dx=\frac{1}{2}\left[\frac{x^2}{2}\right]_1^e=\frac{e^2-1}{4}$$

31.25 13)

$$\int_1^e \frac{\ln^2 x}{x^2}dx = \int_1^e \left(-\frac{1}{x}\right)' \ln^2 x dx = \left[-\frac{\ln^2 x}{x}\right]_1^e - \int_1^e -\frac{1}{x}\left(\ln^2 x\right)' dx =$$

$$= -\frac{\ln^2 e}{e} - \int_1^e -\frac{1}{x} 2\ln x \frac{1}{x} dx = -\frac{1}{e} + 2\int_1^e \frac{1}{x^2}\ln x dx =$$

$$= -\frac{1}{e} + 2\left(\int_1^e \left(-\frac{1}{x}\right)' \ln x dx\right) = -\frac{1}{e} + 2\left(\left[-\frac{\ln x}{x}\right]_1^e - \int_1^e -\frac{1}{x}(\ln x)' dx\right)$$

$$= -\frac{1}{e} + 2\left(-\frac{\ln e}{e} + \int_1^e \frac{1}{x}\frac{1}{x} dx\right) = -\frac{1}{e} - \frac{2}{e} + 2\int_1^e \frac{1}{x^2}dx = -\frac{3}{e} + 2\left[-\frac{1}{x}\right]_1^e =$$

$$= \frac{3}{e} + 2\left(-\frac{1}{e} + 1\right) = \frac{3}{e} - \frac{2}{e} + 2 = -\frac{5}{e} + 2$$

31.25 14)

$$\int_1^e \ln^2 x\, dx = \int_1^e (x)' \ln^2 x\, dx = \left[x\ln^2 x\right]_1^e - \int_1^e x\left(\ln^2 x\right)' dx =$$

$$= e\ln^2 e - \int_1^e \cancel{x}\, 2\ln x \frac{1}{\cancel{x}} dx = e - 2\int_1^e \ln x\, dx = e - 2\int_1^e (x)' \ln x\, dx =$$

$$= e - 2\left(\left[x\ln x\right]_1^e - \int_1^e x(\ln x)' dx\right) = e - 2\left(e\ln e - \int_1^e \cancel{x}\frac{1}{\cancel{x}}dx\right) =$$

$$= e - 2e + 2\int_1^e 1dx = -e + 2\left[1\cdot(e-1)\right] = -e + 2e - 2 = e - 2$$

31.25 15)

Μια αρχική της $\sqrt{x} = x^{\frac{1}{2}}$ είναι η $\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} = \dfrac{2}{3}\sqrt{x^3}$

Οπότε

$$\int_1^e \sqrt{x}\ln^2 x\, dx = \int_1^e \left(\frac{2}{3}\sqrt{x^3}\right)' \ln^2 x dx = \left[\frac{2}{3}\sqrt{x^3}\ln^2 x\right]_1^e - \int_1^e \frac{2}{3}\sqrt{x^3}\left(\ln^2 x\right)' dx =$$

$$= \frac{2}{3}\sqrt{e^3}\ln^2 e - \int_1^e \frac{2}{3}\sqrt{x^3}\, 2\ln x \frac{1}{x}dx = \frac{2}{3}\sqrt{e^3} - \frac{4}{3}\int_1^e \cancel{x}\sqrt{x}\ln x \frac{1}{\cancel{x}}dx =$$

$$= \frac{2}{3}\sqrt{e^3} - \frac{4}{3}\int_1^e \sqrt{x}\ln x dx = \frac{2}{3}\sqrt{e^3} - \frac{4}{3}\int_1^e \left(\frac{2}{3}\sqrt{x^3}\right)' \ln x dx =$$

$$= \frac{2}{3}\sqrt{e^3} - \frac{4}{3}\left(\left[\frac{2}{3}\sqrt{x^3}\ln x\right]_1^e - \int_1^e \frac{2}{3}\sqrt{x^3}(\ln x)' dx\right) =$$

$$= \frac{2}{3}\sqrt{e^3} - \frac{4}{3}\left(\frac{2}{3}\sqrt{e^3}\ln e - \int_1^e \frac{2}{3}\sqrt{x^3}\frac{1}{x}dx\right) =$$

$$=\frac{2}{3}\sqrt{e^3}-\frac{8}{9}\sqrt{e^3}+\frac{8}{9}\int_1^e\sqrt{x^3}\,\frac{1}{x}dx=-\frac{2}{9}\sqrt{e^3}+\frac{8}{9}\int_1^e\cancel{x}\sqrt{x}\,\frac{1}{\cancel{x}}dx=$$

$$=-\frac{2}{9}\sqrt{e^3}+\frac{8}{9}\int_1^e\sqrt{x}dx=-\frac{2}{9}\sqrt{e^3}+\frac{8}{9}\left[\frac{2}{3}\sqrt{x^3}\right]_1^e=-\frac{2\sqrt{e^3}}{9}+\frac{16\sqrt{e^3}}{27}-\frac{16}{27}=$$

$$=\frac{10\sqrt{e^3}-16}{27}$$

31.25 16)

$$\int_1^e\ln^3x\,dx=\int_1^e(x)'\ln^3x\,dx=\left[x\ln^3x\right]_1^e-\int_1^e x\left(\ln^3x\right)'dx=$$

$$=e\ln^3e-\int_1^e\cancel{x}\,3\ln^2x\,\frac{1}{\cancel{x}}dx=e-3\int_1^e\ln^2x\,dx=e-3\int_1^e(x)'\ln^2x\,dx=$$

$$=e-3\left(\left[x\ln^2x\right]_1^e-\int_1^e x\left(\ln^2x\right)'dx\right)=e-3\left(e\ln^2e-\int_1^e\cancel{x}\,2\ln x\,\frac{1}{\cancel{x}}dx\right)=$$

$$=e-3e+6\int_1^e\ln x\,dx=-2e+6\int_1^e(x)'\ln x\,dx=$$

$$=-2e+6\left(\left[x\ln x\right]_1^e-\int_1^e x\left(\ln x\right)'dx\right)=-2e+6\left(e\ln e-\int_1^e\cancel{x}\,\frac{1}{\cancel{x}}dx\right)=$$

$$=-2e+6e-6\int_1^e1dx=4e-6\left[1\cdot(e-1)\right]=4e-6e+6=6-2e$$