Γ ΛΥΚΕΙΟΥ ΜΕΡΟΣ Β

31.23 1)

θέτουμε $y=\sqrt{x+2}\Rightarrow x+2=y^2\Rightarrow x=y^2-2$
$\Rightarrow dx=2ydy$
για $x=-1\Rightarrow y=1$
για $x=2\Rightarrow y=2$

α) $$\int_{-1}^{2}\frac{3x^2}{\sqrt{x+2}}dx = \int_1^2\frac{3(y^2-2)^2}{\cancel{y}}2\cancel{y}dy=$$

$$=6\int_1^2(y^2-2)^2dy=6\int_1^2(y^4-4y^2+4)dy=6\left[\frac{y^5}{5}-4\frac{y^3}{3}+4y\right]_1^2=$$

$$=6\left[\frac{2^5}{5}-4\frac{2^3}{3}+4\cdot 2-\left(\frac{1^5}{5}-4\frac{1^3}{3}+4\cdot 1\right)\right]=\ldots=\frac{26}{5}$$

θέτουμε $y=\sqrt{x-3}\Rightarrow x-3=y^2\Rightarrow x=y^2+3$
$\Rightarrow dx=2ydy$
για $x=3\Rightarrow y=0$
για $x=4\Rightarrow y=1$

β) $$\int_3^4\frac{1}{1+\sqrt{x-3}}dx = \int_0^1\frac{1}{1+y}2ydy=2\int_0^1\frac{y+1-1}{1+y}dy=$$

$$=2\int_0^1\left(\cancel{\frac{y+1}{1+y}}-\frac{1}{1+y}\right)dy=2\int_0^1\left(1-\frac{1}{1+y}\right)dy=2\left[y-\ln|y+1|\right]_0^1=2-2\ln 2$$

31.23 2)

θέτουμε $\boxed{y=\sqrt{1+x}}\Rightarrow x+1=y^2\Rightarrow\boxed{x=y^2-1}\Rightarrow\boxed{dx=2ydy}$
για $x=0\Rightarrow y=1$
για $x=3\Rightarrow y=2$

$$\int_0^3\frac{5x^2}{\sqrt{1+x}}dx = \int_1^2\frac{5(y^2-1)^2}{\cancel{y}}2\cancel{y}dy$$

$$=10\int_1^2(y^2-1)^2dy=10\int_1^2 y^4-2y^2+1dy=10\left[\frac{y^5}{5}-2\frac{y^3}{3}+y\right]_1^2=$$

$$=10\left[\left(\frac{2^5}{5}-2\frac{2^3}{3}+2\right)-\left(\frac{1^5}{5}-2\frac{1^3}{3}+1\right)\right]=10\left(\frac{31}{5}-\frac{14}{3}+1\right)=\frac{76}{3}$$

31.23 3)

θέτουμε $\boxed{y=\sqrt{x}}\Rightarrow x=y^2\Rightarrow\boxed{dx=2ydy}$
για $x=1\Rightarrow y=1$
για $x=9\Rightarrow y=3$

$$\int_1^9\frac{1}{1+\sqrt{x}}dx = \int_1^3\frac{1}{1+y}2ydy=2\int_1^3\frac{y}{1+y}dy=2\int_1^3\frac{y+1-1}{1+y}dy=$$

$$=2\int_1^3\cancel{\frac{y+1}{1+y}}dy-2\int_1^3\frac{1}{1+y}dy=2(3-1)-2\left[\ln|y+1|\right]_1^3=4-2(\ln|3+1|-\ln|1+1|)=$$

$$=4-2(\ln 4-\ln 2)=4-2\ln 2$$

31.23 4)

θέτουμε $\boxed{y=\sqrt{x}}\Rightarrow\boxed{x=y^2}\Rightarrow\boxed{dx=2ydy}$
για $x=1\Rightarrow y=1$
για $x=4\Rightarrow y=2$

$$\int_1^4\frac{x+1}{\sqrt{x^3}}dx = \int_1^2\frac{y^2+1}{y^{\cancel{3}2}}2\cancel{y}dy=2\int_1^2\frac{y^2}{y^2}+\frac{1}{y^2}dy=$$

$$=2\int_1^2\left(1+\frac{1}{y^2}\right)dy=2\left[y-\frac{1}{y}\right]_1^2=2\left[\left(2-\frac{1}{2}\right)-\cancel{\left(1-\frac{1}{1}\right)}\right]=3$$

31.23 5)

$$\int_0^8 \frac{1}{1+\sqrt{x+1}}dx \overset{\substack{\text{θέτουμε } \boxed{y=\sqrt{x+1}} \Rightarrow x+1=y^2 \Rightarrow \boxed{x=y^2-1} \Rightarrow \boxed{dx=2ydy} \\ \text{για } x=0 \Rightarrow y=1 \\ \text{για } x=8 \Rightarrow y=3}}{=} \int_1^3 \frac{1}{1+y}2ydy = 2\int_1^3 \frac{y+1-1}{1+y}dy =$$

$$= 2\int_1^3 \frac{\cancel{y+1}}{\cancel{1+y}}dy - 2\int_1^3 \frac{1}{1+y}dy = 2(3-1) - 2\left[\ln|y+1|\right]_1^3 = 4 - 2\left(\ln|3+1| - \ln|1+1|\right) =$$

$$= 4 - 2(\ln 4 - \ln 2) = 4 - 2\ln 2$$

31.23 6)

Έστω $I = \int_2^3 \frac{x+1}{\sqrt{x-2}+2}dx$. Τότε

$$I = \int_2^3 \frac{x+1}{\sqrt{x-2}+2}dx \overset{\substack{\text{θέτουμε } \boxed{y=\sqrt{x-2}} \Rightarrow x-2=y^2 \Rightarrow \boxed{x=y^2+2} \Rightarrow \boxed{dx=2ydy} \\ \text{για } x=2 \Rightarrow y=0 \\ \text{για } x=3 \Rightarrow y=1}}{=} \int_0^1 \frac{y^2+2+1}{y+2}2ydy = 2\int_0^1 \frac{y^3+3y}{y+2}dy$$

Κάνουμε την διαίρεση των πολυωνύμων

$$\begin{array}{rl|l} \cancel{y^3} & +3y & y+2 \\ \cline{3-3} -\cancel{y^3} & -2y^2 & y^2-2y+7 \\ \cline{1-2} \cancel{-2y^2} & +3y & \\ \cancel{2y^2} & +4y & \\ \cline{1-2} & \cancel{7y} & \\ & -\cancel{6y}-14 & \\ \cline{2-2} & -14 & \end{array}$$

Οπότε $\quad y^3+3y = (y^2-2y+7)(y+2)-14$

Άρα

$$I = 2\int_0^1 \frac{y^3+3y}{y+2}dy = 2\int_0^1 \frac{(y^2-2y+7)(y+2)-14}{y+2}dy =$$

$$= 2\int_0^1 \frac{(y^2-2y+7)\cancel{(y+2)}}{\cancel{y+2}}dy + 2\int_0^1 \frac{-14}{y+2}dy =$$

$$= 2\int_0^1 \frac{(y^2-2y+7)\cancel{(y+2)}}{\cancel{y+2}}dy + 2\int_0^1 \frac{-14}{y+2}dy = 2\left[\frac{y^3}{3} - 2\frac{y^2}{2} + 7y\right]_0^1 - 28\left[\ln|y+2|\right]_0^1 =$$

$$= 2\left(\frac{1^3}{3} - 2\frac{1^2}{2} + 7\cdot 1\right) - 28\left(\ln|1+2| - \ln|0+2|\right) = \frac{38}{3} - 28\ln\frac{3}{2}$$