Γ ΛΥΚΕΙΟΥ ΜΕΡΟΣ Β

31.20 1)

α) Είναι $\frac{4}{x^2-4}=\frac{4}{(x-2)(x+2)}$

Τώρα θα πρέπει να βρούμε τα Α και Β ώστε $\frac{4}{(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x+2}$

Έχουμε $\frac{4}{(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x+2}\Rightarrow$

$$\Rightarrow 4=\cancel{(x-2)}(x+2)\frac{A}{\cancel{x-2}}+(x-2)\cancel{(x+2)}\frac{B}{\cancel{x+2}}\Rightarrow$$

$$\Rightarrow 4=A(x+2)+B(x-2)\Rightarrow 4=Ax+2A+Bx-2B\Rightarrow$$

$$\Rightarrow 4=(A+B)x+(2A-2B)\Rightarrow\begin{cases}A+B=0\\2A-2B=4\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=1\\B=-1\end{cases}$$

Οπότε $\frac{4}{(x-2)(x+2)}=\frac{1}{x-2}-\frac{1}{x+2}$

Άρα $\int_4^6\frac{4}{x^2-4}dx=\int_4^6\frac{1}{x-2}-\frac{1}{x+2}dx=\int_4^6\frac{1}{x-2}dx-\int_4^6\frac{1}{x+2}dx=$

$$=\left[\ln|x-2|\right]_4^6-\left[\ln|x+2|\right]_4^6==\ln|6-2|-\ln|4-2|-\ln|6+2|+\ln|4+2|=$$

$$=\ln4-\ln2-\ln8+\ln6=\ln\frac{4\cdot6}{2\cdot8}=\ln\frac{3}{2}$$

β) Είναι $\frac{3x+1}{x^2-x-6}\overset{\Delta=25,\ x_{1,2}=\frac{1\pm5}{2}=\begin{cases}\nearrow x_1=\frac{1-5}{2}\Rightarrow x_1=-2\\ \searrow x_2=\frac{1+5}{2}\Rightarrow x_2=3\end{cases}}{=}\frac{3x+1}{(x+2)(x-3)}$

Οπότε θα πρέπει να βρούμε τα Α και Β ώστε $\frac{3x+1}{(x+2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-3}$

Έχουμε $\frac{3x+1}{(x+2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-3}\Rightarrow$

$$\Rightarrow 3x+1=\cancel{(x+2)}(x-3)\frac{A}{\cancel{x+2}}+(x+2)\cancel{(x-3)}\frac{B}{\cancel{x-3}}$$

$$\Rightarrow 3x+1=A(x-3)+B(x+2)\Rightarrow 3x+1=Ax-3A+Bx+2B\Rightarrow$$

$$\Rightarrow 3x+1=(A+B)x+(-3A+2B)\Rightarrow\begin{cases}A+B=3\\-3A+2B=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=1\\B=2\end{cases}$$

Οπότε $\frac{3x+1}{(x+2)(x-3)}=\frac{1}{x+2}+\frac{2}{x-3}$

Άρα $\int_0^1\frac{3x+1}{x^2-x-6}dx=\int_0^1\frac{1}{x+2}+\frac{2}{x-3}dx=\int_0^1\frac{1}{x+2}dx+2\int_0^1\frac{1}{x-3}dx=$

$$=\left[\ln|x+2|\right]_0^1+2\left[\ln|x-3|\right]_0^1==\ln|1+2|-\ln|0+2|+2\left(\ln|1-3|-\ln|0-3|\right)=$$

$$=\ln3-\ln2+2\ln2-2\ln3=\ln2-\ln3=\ln\frac{2}{3}$$

31.20 2)

Είναι $\frac{2}{x^2-1}=\frac{2}{(x-1)(x+1)}$

Οπότε θα πρέπει να βρούμε τα A και B ώστε $\frac{2}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$

Έχουμε

$$\frac{2}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}\Rightarrow$$

$$\Rightarrow \cancel{(x-1)(x+1)}\frac{2}{\cancel{(x-1)(x+1)}}=\cancel{(x-1)}(x+1)\frac{A}{\cancel{x-1}}+(x-1)\cancel{(x+1)}\frac{B}{\cancel{x+1}}\Rightarrow$$

$$\Rightarrow 2=A(x+1)+B(x-1)\Rightarrow 2=Ax+A+Bx-B\Rightarrow$$

$$\Rightarrow 2=(A+B)x+A-B\Rightarrow\begin{cases}A+B=0\\A-B=2\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=1\\B=-1\end{cases}$$

Οπότε $\frac{2}{(x-1)(x+1)}=\frac{1}{x-1}-\frac{1}{x+1}$

Άρα

$$\int_2^3\frac{2}{x^2-1}dx=\int_2^3\frac{1}{x-1}-\frac{1}{x+1}dx=\int_2^3\frac{1}{x-1}dx-\int_2^3\frac{1}{x+1}dx=$$

$$=\left[\ln|x-1|\right]_2^3-\left[\ln|x+1|\right]_2^3=\ln|3-1|-\ln|2-1|-\ln|3+1|+\ln|2+1|=$$

$$=\ln2-\ln1-\ln4+\ln3=\ln\frac{2\cdot3}{4}=\ln\frac{3}{2}$$

31.20 3)

Είναι $\frac{1}{x^2-5x+6}\overset{\Delta=1,\ x_{1,2}=\frac{5\pm1}{2}=\begin{cases}x_1=\frac{5-1}{2}\Rightarrow x_1=2\\ x_2=\frac{5+1}{2}\Rightarrow x_2=3\end{cases}}{=}\frac{1}{(x-2)(x-3)}$

Οπότε θα πρέπει να βρούμε τα A και B ώστε $\frac{1}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}$

Έχουμε

$$\frac{1}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}\Rightarrow$$

$$\Rightarrow \cancel{(x-2)(x-3)}\frac{1}{\cancel{(x-2)(x-3)}}=\cancel{(x-2)}(x-3)\frac{A}{\cancel{x-2}}+(x-2)\cancel{(x-3)}\frac{B}{\cancel{x-3}}$$

$$\Rightarrow 1=A(x-3)+B(x-2)\Rightarrow 1=Ax-3A+Bx-2B\Rightarrow$$

$$\Rightarrow 1=(A+B)x+(-3A-2B)\Rightarrow\begin{cases}A+B=0\\-3A-2B=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=-1\\B=1\end{cases}$$

Οπότε $\frac{1}{(x-2)(x-3)}=-\frac{1}{x-2}+\frac{1}{x-3}$

Άρα

$$\int_0^1 \frac{1}{x^2-5x+6}dx = \int_0^1 -\frac{1}{x-2}+\frac{1}{x-3}dx = -\int_0^1 \frac{1}{x-2}dx + \int_0^1 \frac{1}{x-3}dx =$$

$$= -\left[\ln|x-2|\right]_0^1 + \left[\ln|x-3|\right]_0^1 =$$

$$= -\ln|1-2| + \ln|0-2| + \ln|1-3| - \ln|0-3| =$$

$$= \ln 2 + \ln 2 - \ln 3 = \ln\frac{4}{3}$$

31.20 4)

Είναι $\frac{1}{x^2-\alpha^2} = \frac{1}{(x-\alpha)(x+\alpha)}$

Οπότε θα πρέπει να βρούμε τα A και B ώστε $\frac{1}{(x-\alpha)(x+\alpha)} = \frac{A}{x-\alpha}+\frac{B}{x+\alpha}$

Έχουμε

$$\frac{1}{(x-\alpha)(x+\alpha)} = \frac{A}{x-\alpha}+\frac{B}{x+\alpha} \Rightarrow$$

$$\Rightarrow \cancel{(x-\alpha)(x+\alpha)}\frac{1}{\cancel{(x-\alpha)(x+\alpha)}} = \cancel{(x-\alpha)}(x+\alpha)\frac{A}{\cancel{x-\alpha}} + (x-\alpha)\cancel{(x+\alpha)}\frac{B}{\cancel{x+\alpha}} \Rightarrow$$

$$\Rightarrow 1 = A(x+\alpha)+B(x-\alpha) \Rightarrow 1 = Ax+\alpha A+Bx-\alpha B \Rightarrow$$

$$\Rightarrow 1 = (A+B)x+(\alpha A-\alpha B) \Rightarrow \begin{cases} A+B=0 \\ \alpha A-\alpha B=1 \end{cases} \overset{\text{λύνουμε το σύστημα}}{\Rightarrow} \begin{cases} A=\frac{1}{2\alpha} \\ B=-\frac{1}{2\alpha} \end{cases}$$

Οπότε $\frac{1}{(x-\alpha)(x+\alpha)} = \frac{\frac{1}{2\alpha}}{x-\alpha} - \frac{\frac{1}{2\alpha}}{x+\alpha}$

Άρα

$$\int_0^{\frac{\alpha}{2}} \frac{1}{x^2-\alpha^2}dx = \int_0^{\frac{\alpha}{2}} \frac{\frac{1}{2\alpha}}{x-\alpha} - \frac{\frac{1}{2\alpha}}{x+\alpha}dx = \frac{1}{2\alpha}\int_0^{\frac{\alpha}{2}} \frac{1}{x-\alpha}dx - \frac{1}{2\alpha}\int_0^{\frac{\alpha}{2}} \frac{1}{x+\alpha}dx =$$

$$= \frac{1}{2\alpha}\left[\ln|x-\alpha|\right]_0^{\frac{\alpha}{2}} - \frac{1}{2\alpha}\left[\ln|x+\alpha|\right]_0^{\frac{\alpha}{2}} =$$

$$= \frac{1}{2\alpha}\left(\ln\left|\frac{\alpha}{2}-\alpha\right| - \ln|0-\alpha| - \ln\left|\frac{\alpha}{2}+\alpha\right| + \ln|0+\alpha|\right) \overset{\alpha>0}{=}$$

$$= \frac{1}{2\alpha}\left(\ln\frac{\alpha}{2} - \cancel{\ln\alpha} - \ln\frac{3\alpha}{2} + \cancel{\ln\alpha}\right) = \frac{1}{2\alpha}\ln\frac{\frac{\alpha}{\cancel{2}}}{\frac{3\alpha}{\cancel{2}}} = \frac{1}{2\alpha}\ln\frac{1}{3} = -\frac{1}{2\alpha}\ln 3$$

31.20 5)

Είναι $\dfrac{1}{x^2-x-2} \overset{\Delta=9,\ x_{1,2}=\frac{1\pm3}{2}=\begin{cases}x_1=\frac{1-3}{2}\Rightarrow x_1=-1\\ x_2=\frac{1+3}{2}\Rightarrow x_2=2\end{cases}}{=} \dfrac{1}{(x+1)(x-2)}$

Οπότε θα πρέπει να βρούμε τα A και B ώστε $\dfrac{1}{(x+1)(x-2)}=\dfrac{A}{x+1}+\dfrac{B}{x-2}$

Έχουμε

$$\frac{1}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2}\Rightarrow$$

$$\Rightarrow \cancel{(x+1)(x-2)}\frac{1}{\cancel{(x+1)(x-2)}}=\cancel{(x+1)}(x-2)\frac{A}{\cancel{x+1}}+(x+1)\cancel{(x-2)}\frac{B}{\cancel{x-2}}$$

$$\Rightarrow 1=A(x-2)+B(x+1)\Rightarrow 1=Ax-2A+Bx+B\Rightarrow$$

$$\Rightarrow 1=(A+B)x+(-2A+B)\Rightarrow\begin{cases}A+B=0\\-2A+B=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=-\dfrac{1}{3}\\B=\dfrac{1}{3}\end{cases}$$

Οπότε $\dfrac{1}{(x+1)(x-2)}=-\dfrac{\frac{1}{3}}{x+1}+\dfrac{\frac{1}{3}}{x-2}$

Άρα

$$\int_{-2}^{1}\frac{1}{x^2-x-2}dx=\int_{-2}^{1}-\frac{\frac{1}{3}}{x+1}+\frac{\frac{1}{3}}{x-2}dx=-\frac{1}{3}\int_{-2}^{1}\frac{1}{x+1}dx+\frac{1}{3}\int_{-2}^{1}\frac{1}{x-2}dx=$$

$$=-\frac{1}{3}\Big[\ln|x+1|\Big]_{-2}^{1}+\frac{1}{3}\Big[\ln|x-2|\Big]_{-2}^{1}=$$

$$=-\frac{1}{3}\left(\ln|1+1|-\cancel{\ln|-2+1|}\right)+\frac{1}{3}\left(\cancel{\ln|1-2|}-\ln|-2-2|\right)$$

$$=-\frac{1}{3}\ln 2-\frac{1}{3}\ln 4=-\frac{1}{3}(\ln 2+\ln 4)=-\frac{1}{3}\ln 8=-\ln\sqrt[3]{8}=-\ln 2$$

31.20 6)

Είναι $\dfrac{x+1}{x^2-3x+2} \overset{\Delta=1,\ x_{1,2}=\frac{3\pm1}{2}=\begin{cases}x_1=\frac{3-1}{2}\Rightarrow x_1=1\\ x_2=\frac{3+1}{2}\Rightarrow x_2=2\end{cases}}{=} \dfrac{x+1}{(x-1)(x-2)}$

Οπότε θα πρέπει να βρούμε τα A και B ώστε $\dfrac{x+1}{x^2-3x+2}=\dfrac{A}{x-1}+\dfrac{B}{x-2}$

Έχουμε

$$\frac{x+1}{x^2-3x+2}=\frac{A}{x-1}+\frac{B}{x-2}\Rightarrow$$

$$\Rightarrow \cancel{(x-1)(x-2)}\frac{x+1}{\cancel{(x-1)(x-2)}}=\cancel{(x-1)}(x-2)\frac{A}{\cancel{x-1}}+(x-1)\cancel{(x-2)}\frac{B}{\cancel{x-2}}$$

$$\Rightarrow x+1=A(x-2)+B(x-1)\Rightarrow x+1=Ax-2A+Bx-B\Rightarrow$$

$$\Rightarrow x+1=(A+B)x+(-2A-B)\Rightarrow\begin{cases}A+B=1\\-2A-B=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=-2\\B=3\end{cases}$$

Οπότε $\frac{x+1}{x^2-3x+2}=\frac{-2}{x-1}+\frac{3}{x-2}$

Άρα

$$\int_3^4\frac{x+1}{x^2-3x+2}dx=\int_3^4\frac{-2}{x-1}+\frac{3}{x-2}dx=-2\int_3^4\frac{1}{x-1}dx+3\int_3^4\frac{1}{x-2}dx=$$

$$=-2\left[\ln|x-1|\right]_3^4+3\left[\ln|x-2|\right]_3^4=$$

$$=-2\left(\ln|4-1|-\ln|3-1|\right)+3\left(\ln|4-2|-\ln|3-2|\right)=$$

$$=-2(\ln 3-\ln 2)+3(\ln 2-\cancel{\ln 1})=-2\ln 3+2\ln 2+3\ln 2=$$

$$=-2\ln 3+5\ln 2=\ln\frac{32}{9}$$

31.20 7)

Είναι $\frac{x}{x^2-x-12}\overset{\Delta=49,\ x_{1,2}=\frac{1\pm7}{2}=\begin{cases}x_1=\frac{1-7}{2}\Rightarrow x_1=-3\\x_2=\frac{1+7}{2}\Rightarrow x_2=4\end{cases}}{=}\frac{x}{(x+3)(x-4)}$

Οπότε θα πρέπει να βρούμε τα Α και Β ώστε $\frac{x}{x^2-x-12}=\frac{A}{x+3}+\frac{B}{x-4}$

Έχουμε

$$\frac{x}{x^2-x-12}=\frac{A}{x+3}+\frac{B}{x-4}\Rightarrow$$

$$\Rightarrow\cancel{(x+3)(x-4)}\frac{x}{\cancel{(x+3)(x-4)}}=\cancel{(x+3)}(x-4)\frac{A}{\cancel{x+3}}+(x+3)\cancel{(x-4)}\frac{B}{\cancel{x-4}}$$

$$\Rightarrow x=A(x-4)+B(x+3)\Rightarrow x=Ax-4A+Bx+3B\Rightarrow$$

$$\Rightarrow x=(A+B)x+(-4A+3B)\Rightarrow\begin{cases}A+B=1\\-4A+3B=0\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=\frac{3}{7}\\B=\frac{4}{7}\end{cases}$$

Οπότε $\frac{x}{x^2-x-12}=\frac{\frac{3}{7}}{x+3}+\frac{\frac{4}{7}}{x-4}$

Άρα

$$\int_{-2}^{3}\frac{x}{x^2-x-12}dx=\int_{-2}^{3}\frac{\frac{3}{7}}{x+3}+\frac{\frac{4}{7}}{x-4}dx=\frac{3}{7}\int_{-2}^{3}\frac{1}{x+3}dx+\frac{4}{7}\int_{-2}^{3}\frac{1}{x-4}dx=$$

$$=\frac{3}{7}\left[\ln|x+3|\right]_{-2}^{3}+\frac{4}{7}\left[\ln|x-4|\right]_{-2}^{3}=$$

$$=\frac{3}{7}\left(\ln|3+3|-\cancel{\ln|-2+3|}\right)+\frac{4}{7}\left(\cancel{\ln|3-4|}-\ln|-2-4|\right)=$$

$$=\frac{3}{7}\ln 6-\frac{4}{7}\ln 6=-\frac{\ln 6}{7}$$

31.20 8)

Θα πρέπει να βρούμε τα Α και Β ώστε $\frac{\alpha}{(x+\alpha)(x+2\alpha)}=\frac{A}{x+\alpha}+\frac{B}{x+2\alpha}$

Έχουμε

$$\frac{\alpha}{(x+\alpha)(x+2\alpha)}=\frac{A}{x+\alpha}+\frac{B}{x+2\alpha}$$

$$\Rightarrow \cancel{(x+\alpha)(x+2\alpha)}\frac{\alpha}{\cancel{(x+\alpha)(x+2\alpha)}}=\cancel{(x+\alpha)}(x+2\alpha)\frac{A}{\cancel{x+\alpha}}+(x+\alpha)\cancel{(x+2\alpha)}\frac{B}{\cancel{x+2\alpha}}\Rightarrow$$

$$\Rightarrow \alpha=A(x+2\alpha)+B(x+\alpha)\Rightarrow \alpha=Ax+2\alpha A+Bx+\alpha B\Rightarrow$$

$$\Rightarrow \alpha=(A+B)x+(2\alpha A+\alpha B)\Rightarrow \begin{cases}A+B=0\\ 2\alpha A+\alpha B=\alpha\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=1\\ B=-1\end{cases}$$

Οπότε $\frac{\alpha}{(x+\alpha)(x+2\alpha)}=\frac{1}{x+\alpha}-\frac{1}{x+2\alpha}$

Άρα

$$\int_{0}^{\alpha}\frac{\alpha}{(x+\alpha)(x+2\alpha)}dx=\int_{0}^{\alpha}\frac{1}{x+\alpha}-\frac{1}{x+2\alpha}dx=\int_{0}^{\alpha}\frac{1}{x+\alpha}dx-\int_{0}^{\alpha}\frac{1}{x+2\alpha}dx=$$

$$=\left[\ln|x+\alpha|\right]_{0}^{\alpha}-\left[\ln|x+2\alpha|\right]_{0}^{\alpha}=$$

$$=\ln|\alpha+\alpha|-\ln|0+\alpha|-\ln|\alpha+2\alpha|+\ln|0+2\alpha|=$$

$$=\ln|2\alpha|-\ln|\alpha|-\ln|3\alpha|+\ln|2\alpha|=\ln\frac{|2\alpha\cdot 2\alpha|}{|\alpha\cdot 3\alpha|}=\ln\frac{4}{3}$$

31.20 9)

Είναι $\frac{2x+1}{x^2-5x+6}\overset{\Delta=1,\ x_{1,2}=\frac{5\pm 1}{2}=\begin{cases}x_1=\frac{5-1}{2}\Rightarrow x_1=2\\ x_2=\frac{5+1}{2}\Rightarrow x_2=3\end{cases}}{=}\frac{2x+1}{(x-2)(x-3)}$

Οπότε θα πρέπει να βρούμε τα Α και Β ώστε $\frac{2x+1}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}$

Έχουμε

$$\frac{2x+1}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}\Rightarrow$$

$$\Rightarrow \cancel{(x-2)(x-3)}\frac{2x+1}{\cancel{(x-2)(x-3)}}=\cancel{(x-2)}(x-3)\frac{A}{\cancel{x-2}}+(x-2)\cancel{(x-3)}\frac{B}{\cancel{x-3}}$$

$$\Rightarrow 2x+1=A(x-3)+B(x-2)\Rightarrow 2x+1=Ax-3A+Bx-2B\Rightarrow$$

$$\Rightarrow 2x+1=(A+B)x+(-3A-2B)\Rightarrow\begin{cases}A+B=2\\-3A-2B=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=-5\\B=7\end{cases}$$

Οπότε $\frac{2x+1}{(x-2)(x-3)}=\frac{-5}{x-2}+\frac{7}{x-3}$

Άρα

$$\int_{-1}^{1}\frac{2x+1}{x^2-5x+6}dx=\int_{-1}^{1}\frac{-5}{x-2}+\frac{7}{x-3}dx=-5\int_{-1}^{1}\frac{1}{x-2}dx+7\int_{-1}^{1}\frac{1}{x-3}dx=$$

$$=-5\left[\ln|x-2|\right]_{-1}^{1}+7\left[\ln|x-3|\right]_{-1}^{1}=$$

$$=-5\left(\cancel{\ln|1-2|}-\ln|-1-2|\right)+7\left(\ln|1-3|-\ln|-1-3|\right)=$$

$$=5\ln 3+7(\ln 2-\ln 4)=5\ln 3+7(\ln 2-2\ln 2)=5\ln 3-7\ln 2$$

31.20 10)

Είναι $\frac{2x+1}{x^2-5x+4}\overset{\Delta=9,\ x_{1,2}=\frac{5\pm 3}{2}=\begin{cases}\nearrow x_1=\frac{5-3}{2}\Rightarrow x_1=1\\\searrow x_2=\frac{5+3}{2}\Rightarrow x_2=4\end{cases}}{=}\frac{2x+1}{(x-1)(x-4)}$

Οπότε θα πρέπει να βρούμε τα Α και Β ώστε $\frac{2x+1}{(x-1)(x-4)}=\frac{A}{x-1}+\frac{B}{x-4}$

Έχουμε

$$\frac{2x+1}{(x-1)(x-4)}=\frac{A}{x-1}+\frac{B}{x-4}\Rightarrow$$

$$\Rightarrow \cancel{(x-1)(x-4)}\frac{2x+1}{\cancel{(x-1)(x-4)}}=\cancel{(x-1)}(x-4)\frac{A}{\cancel{x-1}}+(x-1)\cancel{(x-4)}\frac{B}{\cancel{x-4}}$$

$$\Rightarrow 2x+1=A(x-4)+B(x-1)\Rightarrow 2x+1=Ax-4A+Bx-B\Rightarrow$$

$$\Rightarrow 2x+1=(A+B)x+(-4A-B)\Rightarrow\begin{cases}A+B=2\\-4A-B=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=-1\\B=3\end{cases}$$

Οπότε $\frac{2x+1}{(x-1)(x-4)}=-\frac{1}{x-1}+\frac{3}{x-4}$

Άρα

$$\int_{-1}^{0}\frac{2x+1}{x^2-5x+4}dx=\int_{-1}^{0}-\frac{1}{x-1}+\frac{3}{x-4}dx=-\int_{-1}^{0}\frac{1}{x-1}dx+3\int_{-1}^{0}\frac{1}{x-4}dx=$$

$$=-\left[\ln|x-1|\right]_{-1}^{0}+3\left[\ln|x-4|\right]_{-1}^{0}=$$

$$=-\cancel{\ln|0-1|}+\ln|-1-1|+3\left(\ln|0-4|-\ln|-1-4|\right)=$$

$$=\ln2+3\ln4-3\ln5=\ln2+6\ln2-3\ln5=7\ln2-3\ln5$$

31.20 11)

Είναι $\frac{x-1}{x^2+3x+2}\overset{\Delta=1\,,\;x_{1,2}=\frac{-3\pm1}{2}=\begin{cases}x_1=\frac{-3-1}{2}\Rightarrow x_1=-2\\ x_2=\frac{-3+1}{2}\Rightarrow x_2=-1\end{cases}}{=}\frac{x-1}{(x+2)(x+1)}$

Οπότε θα πρέπει να βρούμε τα A και B ώστε $\frac{x-1}{(x+2)(x+1)}=\frac{A}{x+2}+\frac{B}{x+1}$

Έχουμε

$$\frac{x-1}{(x+2)(x+1)}=\frac{A}{x+2}+\frac{B}{x+1}\Rightarrow$$

$$\Rightarrow\cancel{(x+2)(x+1)}\frac{x-1}{\cancel{(x+2)(x+1)}}=\cancel{(x+2)}(x+1)\frac{A}{\cancel{x+2}}+(x+2)\cancel{(x+1)}\frac{B}{\cancel{x+1}}$$

$$\Rightarrow x-1=A(x+1)+B(x+2)\Rightarrow x-1=Ax+A+Bx+2B\Rightarrow$$

$$\Rightarrow x-1=(A+B)x+(A+2B)\Rightarrow\begin{cases}A+B=1\\A+2B=-1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=3\\B=-2\end{cases}$$

Οπότε $\frac{x-1}{(x+2)(x+1)}=\frac{3}{x+2}-\frac{2}{x+1}$

Άρα

$$\int_{0}^{1}\frac{x-1}{x^2+3x+2}dx=\int_{0}^{1}\frac{3}{x+2}-\frac{2}{x+1}dx=3\int_{0}^{1}\frac{1}{x+2}dx-2\int_{0}^{1}\frac{1}{x+1}dx=$$

$$=3\left[\ln|x+2|\right]_{0}^{1}-2\left[\ln|x+1|\right]_{0}^{1}=$$

$$=3\left(\ln|1+2|-\ln|0+2|\right)-2\left(\ln|1+1|-\cancel{\ln|0+1|}\right)=$$

$$=3\ln3-3\ln2-2\ln2=3\ln3-5\ln2$$

31.20 12)

Θα πρέπει να βρούμε τα A , B και Γ ώστε

$$\frac{x+1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{\Gamma}{x-3}$$

Έχουμε

$$\frac{x+1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{\Gamma}{x-3}\Rightarrow$$

$$\Rightarrow \cancel{(x-1)(x-2)(x-3)}\frac{x+1}{\cancel{(x-1)(x-2)(x-3)}}=\cancel{(x-1)}(x-2)(x-3)\frac{A}{\cancel{x-1}}+$$

$$+(x-1)\cancel{(x-2)}(x-3)\frac{B}{\cancel{x-2}}+(x-1)(x-2)\cancel{(x-3)}\frac{\Gamma}{\cancel{x-3}}\Rightarrow$$

$$\Rightarrow x+1=A(x-2)(x-3)+B(x-1)(x-3)+\Gamma(x-1)(x-2)\Rightarrow$$

$$\Rightarrow x+1=A\left(x^2-5x+6\right)+B\left(x^2-4x+3\right)+\Gamma\left(x^2-3x+2\right)\Rightarrow$$

$$\Rightarrow x+1=Ax^2-5Ax+6A+Bx^2-4Bx+3B+\Gamma x^2-3\Gamma x+2\Gamma\Rightarrow$$

$$\Rightarrow x+1=(A+B+\Gamma)x^2+(-5A-4B-3\Gamma)x+(6A+3B+2\Gamma)\Rightarrow$$

$$\Rightarrow\begin{cases}A+B+\Gamma=0\\-5A-4B-3\Gamma=1\\6A+3B+2\Gamma=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=1\\B=-3\\\Gamma=2\end{cases}$$

Οπότε $\dfrac{x+1}{(x-1)(x-2)(x-3)}=\dfrac{1}{x-1}-\dfrac{3}{x-2}+\dfrac{2}{x-3}$

Άρα

$$\int_4^5\frac{x+1}{(x-1)(x-2)(x-3)}=\int_4^5\frac{1}{x-1}-\frac{3}{x-2}+\frac{2}{x-3}dx=$$

$$=\int_4^5\frac{1}{x-1}dx-3\int_4^5\frac{1}{x-2}dx+\int_4^5\frac{2}{x-3}dx=$$

$$=\left[\ln|x-1|\right]_4^5-3\left[\ln|x-2|\right]_4^5+2\left[\ln|x-3|\right]_4^5=$$

$$=\left[\ln|5-1|-\ln|4-1|\right]_4^5-3\left[\ln|5-2|-\ln|4-2|\right]_4^5+2\left[\ln|5-3|-\ln|4-3|\right]_4^5=$$

$$=\ln|5-1|-\ln|4-1|-3\left(\ln|5-2|-\ln|4-2|\right)+2\left(\ln|5-3|-\cancel{\ln|4-3|}\right)=$$

$$=\ln4-\ln3-3\ln3+3\ln2+2\ln2=2\ln2-4\ln3+5\ln2=7\ln2-4\ln3$$

31.20 13)

Θα πρέπει να βρούμε τα A , B και Γ ώστε

Έχουμε

$$\frac{1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{\Gamma}{x-3}\Rightarrow$$

$$\Rightarrow \cancel{(x-1)(x-2)(x-3)}\frac{1}{\cancel{(x-1)(x-2)(x-3)}}=\cancel{(x-1)}(x-2)(x-3)\frac{A}{\cancel{x-1}}+$$

$$+(x-1)\cancel{(x-2)}(x-3)\frac{B}{\cancel{x-2}}+(x-1)(x-2)\cancel{(x-3)}\frac{\Gamma}{\cancel{x-3}}\Rightarrow$$

$$\Rightarrow 1=A(x-2)(x-3)+B(x-1)(x-3)+\Gamma(x-1)(x-2)\Rightarrow$$

$$\Rightarrow 1=A\left(x^2-5x+6\right)+B\left(x^2-4x+3\right)+\Gamma\left(x^2-3x+2\right)\Rightarrow$$

$$\Rightarrow 1=Ax^2-5Ax+6A+Bx^2-4Bx+3B+\Gamma x^2-3\Gamma x+2\Gamma\Rightarrow$$

$$\Rightarrow 1=(A+B+\Gamma)x^2+(-5A-4B-3\Gamma)x+(6A+3B+2\Gamma)\Rightarrow$$

$$\Rightarrow\begin{cases}A+B+\Gamma=0\\-5A-4B-3\Gamma=0\\6A+3B+2\Gamma=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=\dfrac{1}{2}\\B=-1\\\Gamma=\dfrac{1}{2}\end{cases}$$

Οπότε $\dfrac{1}{(x-1)(x-2)(x-3)}=\dfrac{\frac{1}{2}}{x-1}-\dfrac{1}{x-2}+\dfrac{\frac{1}{2}}{x-3}$

Άρα

$$\int_{-1}^{0}\frac{1}{(x-1)(x-2)(x-3)}dx=\int_{-1}^{0}\frac{\frac{1}{2}}{x-1}-\frac{1}{x-2}+\frac{\frac{1}{2}}{x-3}dx=$$

$$=\frac{1}{2}\int_{-1}^{0}\frac{1}{x-1}dx-\int_{-1}^{0}\frac{1}{x-2}dx+\frac{1}{2}\int_{-1}^{0}\frac{1}{x-3}dx=$$

$$=\frac{1}{2}\Big[\ln|x-1|\Big]_{-1}^{0}-\Big[\ln|x-2|\Big]_{-1}^{0}+\frac{1}{2}\Big[\ln|x-3|\Big]_{-1}^{0}=$$

$$=\frac{1}{2}\left(\cancel{\ln|0-1|}-\ln|-1-1|\right)-\left(\ln|0-2|-\ln|-1-2|\right)+\frac{1}{2}\left(\ln|0-3|-\ln|-1-3|\right)=$$

$$=-\frac{1}{2}\ln 2-\ln 2+\ln 3+\frac{1}{2}\ln 3-\frac{1}{2}\ln 4=-\frac{3}{2}\ln 2+\frac{3}{2}\ln 3-\ln\sqrt{4}=$$

$$=-\frac{5}{2}\ln 2+\frac{3}{2}\ln 3$$

31.20 14)

Είναι $\dfrac{x^2+x+1}{x^3-x}=\dfrac{x^2+x+1}{x(x-1)(x+1)}$

Οπότε θα πρέπει να βρούμε τα Α , Β και Γ ώστε

$$\frac{x^2+x+1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{\Gamma}{x+1}$$

Έχουμε

$$\frac{x^2+x+1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{\Gamma}{x+1}\Rightarrow$$

$$\Rightarrow \cancel{x(x-1)(x+1)}\frac{x^2+x+1}{\cancel{x(x-1)(x+1)}}=\cancel{x}(x-1)(x+1)\frac{A}{\cancel{x}}+x\cancel{(x-1)}(x+1)\frac{B}{\cancel{x-1}}+$$

$$+x(x-1)\cancel{(x+1)}\frac{\Gamma}{\cancel{x+1}}\Rightarrow$$

$$\Rightarrow x^2+x+1=A(x-1)(x+1)+Bx(x+1)+\Gamma x(x-1)\Rightarrow$$

$$\Rightarrow x^2+x+1=A(x^2-1)+B(x^2+x)+\Gamma(x^2-x)\Rightarrow$$

$$\Rightarrow x^2+x+1=Ax^2-A+Bx^2+Bx+\Gamma x^2-\Gamma x\Rightarrow$$

$$\Rightarrow x^2+x+1=(A+B+\Gamma)x^2+(B-\Gamma)x-A\Rightarrow$$

$$\Rightarrow\begin{cases}A+B+\Gamma=1\\ B-\Gamma=1\\ -A=1\end{cases}\overset{\text{λύνουμε το σύστημα}}{\Rightarrow}\begin{cases}A=-1\\ B=\dfrac{3}{2}\\ \Gamma=\dfrac{1}{2}\end{cases}$$

Οπότε $\dfrac{x^2+x+1}{x(x-1)(x+1)}=-\dfrac{1}{x}+\dfrac{\frac{3}{2}}{x-1}+\dfrac{\frac{1}{2}}{x+1}$

Άρα

$$\int_2^4\frac{x^2+x+1}{x^3-x}dx=\int_2^4-\frac{1}{x}+\frac{\frac{3}{2}}{x-1}+\frac{\frac{1}{2}}{x+1}dx=$$

$$=-\int_2^4\frac{1}{x}dx+\frac{3}{2}\int_2^4\frac{1}{x-1}dx+\frac{1}{2}\int_2^4\frac{1}{x+1}dx=$$

$$=-\left[\ln|x|\right]_2^4+\frac{3}{2}\left[\ln|x-1|\right]_2^4+\frac{1}{2}\left[\ln|x+1|\right]_2^4=$$

$$=-\left(\ln|4|-\ln|2|\right)+\frac{3}{2}\left(\ln|4-1|-\cancel{\ln|2-1|}\right)+\frac{1}{2}\left(\ln|4+1|-\ln|2+1|\right)=$$

$$= -\ln 4 + \ln 2 + \frac{3}{2}\ln 3 + \frac{1}{2}\ln 5 - \frac{1}{2}\ln 3 = -2\ln 2 + \ln 2 + \ln 3 + \frac{1}{2}\ln 5 =$$

$$= -\ln 2 + \ln 3 + \frac{1}{2}\ln 5$$