Γ ΛΥΚΕΙΟΥ ΜΕΡΟΣ Β

31.17 1)

α) $$\int_1^{e^2} x\ln x dx=\int_1^{e^2}\left(\frac{x^2}{2}\right)'\ln x dx=\left[\frac{x^2\ln x}{2}\right]_1^{e^2}-\int_1^{e^2}\frac{x^2}{2}(\ln x)'dx=$$

$$=\frac{2e^4}{2}-\int_1^{e^2}\frac{x^{\cancel{2}}}{2}\frac{1}{\cancel{x}}dx=e^4-\frac{1}{2}\int_1^{e^2}xdx=e^4-\frac{1}{2}\left[\frac{x^2}{2}\right]_1^{e^2}=e^4-\frac{1}{2}\left(\frac{e^4}{2}-\frac{1^2}{2}\right)=\frac{3e^4+1}{4}$$

β) $$\int_1^2 \ln x dx=\int_1^2 (x)'\ln x dx=[x\ln x]_1^2-\int_1^2 x(\ln x)'dx=$$

$$=2\ln 2-\int_1^2 \cancel{x}\frac{1}{\cancel{x}}dx=2\ln 2-\int_1^2 1dx=2\ln 2-1(2-1)=2\ln 2-1$$

γ) $$\int_1^{\frac{1}{e}}\frac{\ln x}{x^2}dx=\int_1^{\frac{1}{e}}\left(-\frac{1}{x}\right)'\ln x dx=\left[-\frac{\ln x}{x}\right]_1^{\frac{1}{e}}-\int_1^{\frac{1}{e}}\left(-\frac{1}{x}\right)(\ln x)'dx=$$

$$=e+\int_1^{\frac{1}{e}}\frac{1}{x}\cdot\frac{1}{x}dx=e+\int_1^{\frac{1}{e}}\frac{1}{x^2}dx=e+\left[-\frac{1}{x}\right]_1^{\frac{1}{e}}=e+\left(-e+\frac{1}{1}\right)=1$$

31.17 2)

$$\int_1^e x\ln x dx=\int_1^e\left(\frac{x^2}{2}\right)'\ln x dx=\left[\frac{x^2\ln x}{2}\right]_1^e-\int_1^e\frac{x^2}{2}(\ln x)'dx=\frac{e^2}{2}-\int_1^e\frac{x^{\cancel{2}}}{2}\frac{1}{\cancel{x}}dx=$$

$$=\frac{e^2}{2}-\frac{1}{2}\int_1^e xdx=\frac{e^2}{2}-\frac{1}{2}\left[\frac{x^2}{2}\right]_1^e=\frac{e^2}{2}-\frac{1}{2}\left(\frac{e^2}{2}-\frac{1^2}{2}\right)=\frac{e^2}{2}-\frac{e^2}{4}+\frac{1}{4}=\frac{e^2+1}{4}$$

31.17 3)

$$\int_{\frac{8}{3}}^1(3x-4)\ln x dx=\int_{\frac{8}{3}}^1\left(\frac{3x^2}{2}-4x\right)'\ln x dx=$$

$$=\left[\left(\frac{3x^2}{2}-4x\right)\ln x\right]_{\frac{8}{3}}^1-\int_{\frac{8}{3}}^1\left(\frac{3x^2}{2}-4x\right)(\ln x)'dx=0-\int_{\frac{8}{3}}^1\left(\frac{3x^2}{2}-4x\right)\frac{1}{x}dx=$$

$$=-\int_{\frac{8}{3}}^1\left(\frac{3x}{2}-4\right)dx=-\left[\frac{3x^2}{4}-4x\right]_{\frac{8}{3}}^1=-\left[\left(\frac{3}{4}-4\right)-\left(\frac{3\frac{64}{9}}{4}-4\frac{8}{3}\right)\right]=$$

$$=-\left[-\frac{13}{4}-\left(\frac{16}{3}-\frac{32}{3}\right)\right]=-\left(-\frac{13}{4}+\frac{16}{3}\right)=-\frac{25}{12}$$

31.17 4)

$$\int_1^{\frac{1}{2}}(4x-1)\ln x dx=\int_1^{\frac{1}{2}}(2x^2-x)'\ln x dx=$$

$$=\left[\left(2x^2-x\right)\ln x\right]_1^{\frac{1}{2}}-\int_1^{\frac{1}{2}}\left(2x^2-x\right)\left(\ln x\right)'dx=0-\int_1^{\frac{1}{2}}\left(2x^2-x\right)\frac{1}{x}dx=-\int_1^{\frac{1}{2}}2x-1dx=$$

$$=-\left[x^2-x\right]_1^{\frac{1}{2}}=-\left[\left(\frac{1}{4}-\frac{1}{2}\right)-\left(1^2-1\right)\right]=\frac{1}{4}$$

31.17 5)

$$\int_1^e \ln x dx=\int_1^e (x)'\ln x dx=\left[x\ln x\right]_1^e-\int_1^e x\left(\ln x\right)'dx=$$

$$=e-\int_1^e \cancel{x}\frac{1}{\cancel{x}}dx=e-\int_1^e 1dx=e-1\left(e-1\right)=e-e+1=1$$

31.17 6)

$$\int_0^1 \ln\left(x+2\right)dx=\int_0^1 (x)'\ln\left(x+2\right)dx=\left[x\ln\left(x+2\right)\right]_0^1-\int_0^1 x\left[\ln\left(x+2\right)\right]'dx=$$

$$=\ln 3-\int_0^1\frac{x}{x+2}dx=\ln 3-\int_0^1\frac{x+2-2}{x+2}dx=\ln 3-\int_0^1 1-\frac{2}{x+2}dx=$$

$$=\ln 3-\int_0^1 1dx+2\int_0^1\frac{1}{x+2}dx=\ln 3-1\left(1-0\right)+2\left[\ln\left(x+2\right)\right]_0^1=$$

$$=\ln 3-1+2\ln\frac{3}{2}=\ln 3-1+\ln\frac{9}{4}=-1+\ln\frac{27}{4}$$

31.17 7)

$$\int_1^2 x\ln\left(x+1\right)dx=\int_1^2\left(\frac{x^2}{2}\right)'\ln\left(x+1\right)dx=\left[\frac{x^2\ln\left(x+1\right)}{2}\right]_1^2-\int_1^2\frac{x^2}{2}\left[\ln\left(x+1\right)\right]'dx=$$

$$=\frac{4\ln 3-\ln 2}{2}-\frac{1}{2}\int_1^2\frac{x^2}{x+1}dx=\frac{4\ln 3-\ln 2}{2}-\frac{1}{2}\int_1^2\frac{x^2-1+1}{x+1}dx=$$

$$=\frac{4\ln 3-\ln 2}{2}-\frac{1}{2}\int_1^2\frac{x^2-1}{x+1}+\frac{1}{x+1}dx=$$

$$=\frac{4\ln 3-\ln 2}{2}-\frac{1}{2}\int_1^2\frac{(x-1)\,\cancel{(x+1)}}{\cancel{x+1}}dx-\frac{1}{2}\int_1^2\frac{1}{x+1}dx=$$

$$=\frac{4\ln 3-\ln 2}{2}-\frac{1}{2}\left[\frac{x^2}{2}-x\right]_1^2-\frac{1}{2}\left[\ln\left(x+1\right)\right]_1^2=$$

$$=\frac{4\ln 3-\ln 2}{2}-\frac{1}{2}\frac{1}{2}-\frac{\ln 3-\ln 2}{2}=\frac{3\ln 3}{2}-\frac{1}{4}=\frac{6\ln 3-1}{4}$$

31.17 8)

$$\int_1^{\sqrt[3]{e}} x^2\ln x\,dx=\int_1^{\sqrt[3]{e}}\left(\frac{x^3}{3}\right)'\ln x\,dx=\left[\frac{x^3\ln x}{3}\right]_1^{\sqrt[3]{e}}-\int_1^{\sqrt[3]{e}}\frac{x^3}{3}\left(\ln x\right)'dx=$$

$$=\frac{\sqrt[3]{e}^3\ln\sqrt[3]{e}}{3}-\int_1^{\sqrt[3]{e}}\frac{x^{\cancel{3}2}}{3}\frac{1}{\cancel{x}}dx\overset{\ln\sqrt[3]{e}=\frac{1}{3}}{=}\frac{e}{9}-\frac{1}{3}\int_1^{\sqrt[3]{e}}x^2dx=\frac{e}{9}-\frac{1}{3}\left[\frac{x^3}{3}\right]_1^{\sqrt[3]{e}}=$$

$$=\frac{e}{9}-\frac{1}{3}\left(\frac{\sqrt[3]{e}^3}{3}-\frac{1^3}{3}\right)=\frac{e}{9}-\frac{e-1}{9}=\frac{1}{9}$$

31.17 9)

$$\int_1^2 x^2\ln x\,dx=\int_1^2\left(\frac{x^3}{3}\right)'\ln x\,dx=\left[\frac{x^3\ln x}{3}\right]_1^2-\int_1^2\frac{x^3}{3}(\ln x)'\,dx=$$

$$=\frac{2^3\ln 2}{3}-\int_1^2\frac{x^{\cancel{3}2}}{3}\frac{1}{\cancel{x}}dx=\frac{8\ln 2}{3}-\frac{1}{3}\int_1^2 x^2\,dx=\frac{8\ln 2}{3}-\frac{1}{3}\left[\frac{x^3}{3}\right]_1^2=$$

$$=\frac{8\ln 2}{3}-\frac{1}{3}\left(\frac{2^3}{3}-\frac{1^3}{3}\right)=\frac{8\ln 2}{3}-\frac{7}{9}=\frac{24\ln 2-7}{9}$$

31.17 10)

$$\int_1^2\left(3x^2-2x-2\right)\ln x\,dx=\int_1^2\left(x^3-x^2-2x\right)'\ln x\,dx=$$

$$=\left[\left(x^3-x^2-2x\right)\ln x\right]_1^2-\int_1^2\left(x^3-x^2-2x\right)(\ln x)'\,dx=$$

$$=0-0-\int_1^2\left(x^3-x^2-2x\right)\frac{1}{x}dx=-\int_1^2\frac{\cancel{x}\left(x^2-x-2\right)}{\cancel{x}}dx=$$

$$=-\left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_1^2=-\left[\left(\frac{2^3}{3}-\frac{2^2}{2}-4\right)-\left(\frac{1^3}{3}-\frac{1^2}{2}-2\right)\right]=-\left(\frac{7}{3}+\frac{1}{2}-4\right)=\frac{7}{6}$$

31.17 11)

$$\int_1^e x^3\ln x\,dx=\int_1^e\left(\frac{x^4}{4}\right)'\ln x dx=\left[\frac{x^4\ln x}{4}\right]_1^e-\int_1^e\frac{x^4}{4}(\ln x)'\,dx=$$

$$=\frac{e^4}{4}-\int_1^e\frac{x^{\cancel{4}3}}{4}\frac{1}{\cancel{x}}dx=\frac{e^4}{4}-\frac{1}{4}\int_1^e x^3dx=\frac{e^4}{4}-\frac{1}{4}\left[\frac{x^4}{4}\right]_1^e=\frac{e^4}{4}-\frac{1}{4}\left(\frac{e^4}{4}-\frac{1^2}{4}\right)=$$

$$=\frac{3e^4+1}{16}$$

31.17 12)

$$\int_1^{\sqrt[5]{e}}x^4\ln x dx=\int_1^{\sqrt[5]{e}}\left(\frac{x^5}{5}\right)'\ln x\,dx=\left[\frac{x^5\ln x}{5}\right]_1^{\sqrt[5]{e}}-\int_1^{\sqrt[5]{e}}\frac{x^5}{5}(\ln x)'\,dx=$$

$$=\frac{\sqrt[5]{e}^5\ln\sqrt[5]{e}}{5}-\int_1^{\sqrt[5]{e}}\frac{x^{\cancel{5}4}}{5}\frac{1}{\cancel{x}}dx\overset{\ln\sqrt[5]{e}=\frac{1}{5}}{=}\frac{e}{25}-\frac{1}{5}\int_1^{\sqrt[5]{e}}x^4\,dx=\frac{e}{25}-\frac{1}{5}\left[\frac{x^5}{5}\right]_1^{\sqrt[5]{e}}=$$

$$=\frac{e}{25}-\frac{1}{5}\left(\frac{e}{5}-\frac{1^5}{5}\right)=\frac{e}{25}-\frac{e}{25}+\frac{1}{25}=\frac{1}{25}$$

31.17 13)

$$\int_1^{\frac{1}{e}}\frac{\ln x}{x^2}dx=\int_1^{\frac{1}{e}}\left(-\frac{1}{x}\right)'\ln x dx=\left[-\frac{\ln x}{x}\right]_1^{\frac{1}{e}}-\int_1^{\frac{1}{e}}\left(-\frac{1}{x}\right)(\ln x)'dx=$$

$$=-\frac{\ln\frac{1}{e}}{\frac{1}{e}}+\int_1^{\frac{1}{e}}\frac{1}{x}\cdot\frac{1}{x}dx\overset{\ln\frac{1}{e}=-1}{=}e+\int_1^{\frac{1}{e}}\frac{1}{x^2}dx=e+\left[-\frac{1}{x}\right]_1^{\frac{1}{e}}=e+\left(-\frac{1}{\frac{1}{e}}+\frac{1}{1}\right)=e+1-e=1$$

31.17 14)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\ln\left(\frac{1-x}{1+x}\right)dx=\int_{-\frac{1}{2}}^{\frac{1}{2}}(x)'\ln\left(\frac{1-x}{1+x}\right)dx=$$

$$=\left[x\ln\left(\frac{1-x}{1+x}\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}-\int_{-\frac{1}{2}}^{\frac{1}{2}}x\left[\ln\left(\frac{1-x}{1+x}\right)\right]'dx=$$

$$=\frac{1}{2}\ln\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)+\frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)-\int_{-\frac{1}{2}}^{\frac{1}{2}}x\frac{\frac{-(1+x)-(1-x)}{(1+x)^{\cancel{2}}}}{\frac{1-x}{\cancel{1+x}}}dx=$$

$$=\frac{1}{2}\ln\frac{\frac{1}{\cancel{2}}}{\frac{3}{\cancel{2}}}+\frac{1}{2}\ln\frac{\frac{3}{\cancel{2}}}{\frac{1}{\cancel{2}}}-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{-2x}{(1-x)(1+x)}dx=\frac{1}{2}\left(\ln\frac{1}{3}+\ln 3\right)-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{-2x}{1-x^2}dx=$$

$$=0-\left[\ln\left(1-x^2\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}=\ln\left(1-\frac{1}{4}\right)-\ln\left(1-\frac{1}{4}\right)=0$$

31.17 15)

$$\int_1^2\ln\left(1+\frac{2}{x}\right)dx=\int_1^2(x)'\ln\left(1+\frac{2}{x}\right)dx=\left[x\ln\left(1+\frac{2}{x}\right)\right]_1^2-\int_1^2 x\left[\ln\left(1+\frac{2}{x}\right)\right]'dx=$$

$$=2\ln 2-\ln 3-\int_1^2 x\frac{-\frac{2}{x^2}}{1+\frac{2}{x}}dx=2\ln 2-\ln 3-\int_1^2 x\frac{-\frac{2}{x^{\cancel{2}}}}{\frac{x+2}{\cancel{x}}}dx=$$

$$=\ln 4-\ln 3-\int_1^2\frac{-2\cancel{x}}{\cancel{x}(x+2)}dx=\ln\frac{4}{3}+2\int_1^2\frac{1}{x+2}dx=$$

$$=\ln\frac{4}{3}+2\left[\ln(x+2)\right]_1^2=\ln\frac{4}{3}+2(\ln 4-\ln 3)=\ln\frac{4}{3}+\ln 16-\ln 9=$$

$$=\ln\frac{4}{3}+\ln\frac{16}{9}=\ln\frac{64}{27}$$

31.17 16)

$$\int_1^e\ln^2 x dx=\int_1^e(x)'\ln^2 x dx=\left[x\ln^2 x\right]_1^e-\int_1^e x\left(\ln^2 x\right)'dx=$$

$$=e-\int_1^e\cancel{x}\frac{2\ln x}{\cancel{x}}dx=e-\int_1^e 2\ln x dx=e-\int_1^e(2x)'\ln x dx=$$

$$=e-\left(\left[2x\ln x\right]_1^e-\int_1^e 2x(\ln x)'dx\right)=e-\left(2e-\int_1^e 2\cancel{x}\frac{1}{\cancel{x}}dx\right)=e-2e+\int_1^e 2dx=$$

$$=-e+2(e-1)=-e+2e-2=e-2$$