35.94

α) $\int_0^1 [f'(x)]^2 dx + \int_0^1 [f(x)]^2 dx = f^2(1) - 1 \Rightarrow$

$$\Rightarrow \int_0^1 [f'(x)]^2 + [f(x)]^2 dx = f^2(1) - 1 \overset{\text{αφαιρούμε το } \int_0^1 2f(x)f'(x)dx}{\Rightarrow}$$

$$\Rightarrow \int_0^1 [f'(x)]^2 + [f(x)]^2 dx - \int_0^1 2f(x)f'(x)dx = f^2(1) - 1 - \int_0^1 2f(x)f'(x)dx \Rightarrow$$

$$\Rightarrow \int_0^1 [f'(x)]^2 - 2f(x)f'(x) + [f(x)]^2 dx = f^2(1) - 1 - \int_0^1 [f^2(x)]' dx \Rightarrow$$

$$\Rightarrow \int_0^1 [f'(x) - f(x)]^2 dx = f^2(1) - 1 - [f^2(x)]_0^1 \Rightarrow$$

$$\Rightarrow \int_0^1 [f'(x) - f(x)]^2 dx = \cancel{f^2(1)} - 1 - \cancel{f^2(1)} + f^2(0) \overset{f(0)=1}{\Rightarrow}$$

$$\Rightarrow \int_0^1 [f'(x) - f(x)]^2 dx = 0 \quad (1)$$

και επειδή $[f'(x) - f(x)]^2 \geq 0$, για κάθε $x \in [0,1]$

θα είναι $[f'(x) - f(x)]^2 = 0 \quad (2)$

διότι αν η $[f'(x) - f(x)]^2$ δεν ήταν παντού ίση με μηδέν θα ίσχυε
$\int_0^1 [f'(x) - f(x)]^2 dx > 0$ <u>άτοπο</u> λόγω της (1)

Οπότε

$(2) \Rightarrow f'(x) - f(x) = 0 \Rightarrow f'(x) = f(x) \Rightarrow f(x) = ce^x \quad (3)$, c: σταθερά

Όμως $(3) \overset{\text{θέτουμε } x=0}{\Rightarrow} f(0) = ce^0 \overset{f(0)=1}{\Rightarrow} c = 1 \quad (4)$

Επομένως $(3) \overset{(4)}{\Rightarrow} f(x) = e^x$