5.38

Θέτουμε $h(x)=\dfrac{f(x)-2x}{x-1}$ οπότε $\lim\limits_{x\to1}h(x)=10$ και ακόμη

$$h(x)=\frac{f(x)-2x}{x-1}\Rightarrow h(x)(x-1)=f(x)-2x\Rightarrow f(x)=h(x)(x-1)+2x$$

Οπότε

$$\lim_{x\to0}\frac{f(x+1)-x^2-2}{3x}\overset{\substack{\text{θέτουμε } y=x+1\\ \text{οπότε } x=y-1\\ \text{όταν } x\to0,\, y\to1}}{=}\lim_{y\to1}\frac{f(y)-(y-1)^2-2}{3(y-1)}\overset{f(x)=h(x)(x-1)+2x}{=}$$

$$=\lim_{y\to1}\frac{h(y)(y-1)+2y-\left(y^2-2y+1\right)^2-2}{3(y-1)}=\lim_{y\to1}\frac{h(y)(y-1)+2y-y^2+2y-1-2}{3(y-1)}=$$

$$=\lim_{y\to1}\frac{h(y)(y-1)-y^2+4y-3}{3(y-1)}=\lim_{y\to1}\frac{h(y)(y-1)-\left(y^2-4y+3\right)}{3(y-1)}\overset{y^2-4y+3=(y-1)(y-3)}{=}$$

$$=\lim_{y\to1}\frac{h(y)(y-1)-(y-1)(y-3)}{3(y-1)}=\lim_{y\to1}\frac{\cancel{(y-1)}\left[h(y)-(y-3)\right]}{3\cancel{(y-1)}}\overset{\lim\limits_{x\to1}h(x)=9}{=}$$

$$=\frac{10-(1-3)}{3}=\boxed{4}$$