5.32 1)

Θέτουμε $h(x)=\frac{f(x)}{x-1}$ και $\varphi(x)=g(x)(x^2+x-2)$

οπότε $\lim_{x\to1}h(x)=-3$ (1) , $\lim_{x\to1}\varphi(x)=4$ (2) , και ακόμη

$$\left.\begin{aligned}h(x)=\frac{f(x)}{x-1}\Rightarrow f(x)=(x-1)h(x)\\ \varphi(x)=g(x)(x^2+x-2)\Rightarrow g(x)=\frac{\varphi(x)}{x^2+x-2}\end{aligned}\right\}\Rightarrow f(x)\cdot g(x)=(x-1)h(x)\frac{\varphi(x)}{x^2+x-2}\Rightarrow$$

$$\overset{x^2+x-2=(x-1)(x+2)}{\Rightarrow} f(x)\cdot g(x)=\cancel{(x-1)}h(x)\frac{\varphi(x)}{\cancel{(x-1)}(x+2)}\Rightarrow f(x)\cdot g(x)=\frac{h(x)\varphi(x)}{x+2}\Rightarrow$$

$$\Rightarrow\lim_{x\to1}\left[f(x)\cdot g(x)\right]=\frac{\lim_{x\to1}h(x)\cdot\lim_{x\to1}\varphi(x)}{\lim_{x\to1}(x+2)}\overset{(1),(2)}{\Rightarrow}\lim_{x\to1}\left[f(x)\cdot g(x)\right]=\frac{-3\cdot4}{3}\Rightarrow$$

$$\Rightarrow\boxed{\lim_{x\to1}\left[f(x)\cdot g(x)\right]=-4}$$

5.32 2)

Θέτουμε $h(x)=\frac{f(x)}{x+1}$ και $\varphi(x)=\frac{g(x)}{x^2+3x+2}$

οπότε $\lim_{x\to-1}h(x)=6$ (1) , $\lim_{x\to1}\varphi(x)=3$ (2) , και ακόμη

$$\left.\begin{aligned}h(x)=\frac{f(x)}{x-1}\Rightarrow f(x)=(x+1)h(x)\\ \varphi(x)=\frac{g(x)}{x^2+3x+2}\Rightarrow g(x)=\varphi(x)(x^2+3x+2)\end{aligned}\right\}\Rightarrow\frac{f(x)}{g(x)}=\frac{(x+1)h(x)}{\varphi(x)(x^2+3x+2)}\Rightarrow$$

$$\overset{x^2+3x+2=(x+1)(x+2)}{\Rightarrow}\frac{f(x)}{g(x)}=\frac{\cancel{(x+1)}h(x)}{\varphi(x)\cancel{(x+1)}(x+2)}\Rightarrow\frac{f(x)}{g(x)}=\frac{h(x)}{\varphi(x)(x+2)}\Rightarrow$$

$$\lim_{x\to-1}\frac{f(x)}{g(x)}=\frac{\lim_{x\to-1}h(x)}{\lim_{x\to-1}\varphi(x)\cdot\lim_{x\to-1}(x+2)}\overset{(1),(2)}{\Rightarrow}\lim_{x\to-1}\frac{f(x)}{g(x)}=\frac{6}{3(-1+2)}\Rightarrow\boxed{\lim_{x\to-1}\frac{f(x)}{g(x)}=2}$$

5.32 3)

Θέτουμε $h(x)=\frac{f(x)}{x^2-4}$ και $\varphi(x)=g(x)(x^3-8)$

οπότε $\lim_{x\to2}h(x)=1$ (1) , $\lim_{x\to1}\varphi(x)=-6$ (2) , και ακόμη

$$\left.\begin{aligned}h(x)=\frac{f(x)}{x^2-4}\Rightarrow f(x)=(x^2-4)h(x)\\ \varphi(x)=g(x)(x^3-8)\Rightarrow g(x)=\frac{\varphi(x)}{x^3-8}\end{aligned}\right\}\Rightarrow f(x)\cdot g(x)=(x^2-4)h(x)\frac{\varphi(x)}{x^3-8}\Rightarrow$$

$$\Rightarrow f(x)\cdot g(x)=\cancel{(x-2)}(x+2)h(x)\cdot\frac{\varphi(x)}{\cancel{(x-2)}(x^2+2x+4)}\Rightarrow$$

$$\Rightarrow f(x)\cdot g(x)=\frac{(x+2)h(x)\varphi(x)}{x^2+2x+4}\Rightarrow$$

$$\Rightarrow \lim_{x\to 2} f(x)g(x)=\frac{\lim_{x\to 2}(x+2)\cdot\lim_{x\to 2}h(x)\cdot\lim_{x\to 2}\varphi(x)}{\lim_{x\to 2}(x^2+2x+4)}\Rightarrow$$

$$\overset{(1),(2)}{\Rightarrow} \lim_{x\to 2} f(x)g(x)=\frac{(2+2)\cdot 1\cdot(-6)}{2^2+2\cdot 2+4}\Rightarrow \boxed{\lim_{x\to 2} f(x)g(x)=-2}$$

5.32 4)

Θέτουμε $h(x)=\dfrac{f(x)}{\sqrt{x+1}-2}$ και $\varphi(x)=g(x)\cdot\left(\sqrt{3x}-3\right)$

οπότε $\lim_{x\to 3}h(x)=4$ (1) , $\lim_{x\to 3}\varphi(x)=5$ (2) , και ακόμη

$$\left.\begin{aligned} h(x)=\frac{f(x)}{\sqrt{x+1}-2}\Rightarrow f(x)=\left(\sqrt{x+1}-2\right)h(x)\\ \varphi(x)=g(x)\cdot\left(\sqrt{3x}-3\right)\Rightarrow g(x)=\frac{\varphi(x)}{\sqrt{3x}-3}\end{aligned}\right\}\Rightarrow f(x)\cdot g(x)=\left(\sqrt{x+1}-2\right)h(x)\frac{\varphi(x)}{\sqrt{3x}-3}\Rightarrow$$

$$\overset{\substack{\text{πολλαπλασιάζουμε αριθμητή}\\ \text{και παρονομαστή με}\\ \left(\sqrt{x+1}+2\right)\left(\sqrt{3x}+3\right)}}{\Rightarrow} f(x)\cdot g(x)=h(x)\varphi(x)\frac{\left(\sqrt{x+1}-2\right)\left(\sqrt{x+1}+2\right)\left(\sqrt{3x}+3\right)}{\left(\sqrt{3x}-3\right)\left(\sqrt{x+1}+2\right)\left(\sqrt{3x}+3\right)}\Rightarrow$$

$$\Rightarrow f(x)\cdot g(x)=h(x)\varphi(x)\frac{\left(\sqrt{x+1}^2-2^2\right)\left(\sqrt{3x}+3\right)}{\left(\sqrt{3x}^2-3^2\right)\left(\sqrt{x+1}+2\right)}\Rightarrow$$

$$\Rightarrow f(x)\cdot g(x)=h(x)\varphi(x)\frac{(x+1-4)\left(\sqrt{3x}+3\right)}{(3x-9)\left(\sqrt{x+1}+2\right)}\Rightarrow$$

$$\Rightarrow f(x)\cdot g(x)=h(x)\varphi(x)\frac{\cancel{(x-3)}\left(\sqrt{3x}+3\right)}{3\cancel{(x-3)}\left(\sqrt{x+1}+2\right)}\Rightarrow$$

$$\Rightarrow \lim_{x\to 3}\left[f(x)\cdot g(x)\right]=\lim_{x\to 3}h(x)\lim_{x\to 3}\varphi(x)\lim_{x\to 3}\frac{\sqrt{3x}+3}{3\left(\sqrt{x+1}+2\right)}\overset{(1),(2)}{\Rightarrow}$$

$$\Rightarrow \lim_{x\to 3}\left[f(x)\cdot g(x)\right]=4\cdot 5\cdot\frac{\sqrt{3\cdot 3}+3}{3\left(\sqrt{3+1}+2\right)}\Rightarrow \boxed{\lim_{x\to 3}\left[f(x)\cdot g(x)\right]=10}$$

5.32 5)

Θέτουμε $h(x)=\dfrac{f(x)}{\sqrt{1+x}-\sqrt{1-x}}$ και $\varphi(x)=\dfrac{g(x)}{\sqrt{2x+1}-1}$

οπότε $\lim_{x\to 0}h(x)=1$ (1) , $\lim_{x\to 0}\varphi(x)=\dfrac{1}{2}$ (2) , και ακόμη

$$\left.\begin{aligned} h(x)=\frac{f(x)}{\sqrt{1+x}-\sqrt{1-x}}\Rightarrow f(x)=\left(\sqrt{1+x}-\sqrt{1-x}\right)h(x)\\ \varphi(x)=\frac{g(x)}{\sqrt{2x+1}-1}\Rightarrow g(x)=\varphi(x)\left(\sqrt{2x+1}-1\right)\end{aligned}\right\}\Rightarrow \frac{f(x)}{g(x)}=\frac{h(x)\left(\sqrt{1+x}-\sqrt{1-x}\right)}{\varphi(x)\left(\sqrt{2x+1}-1\right)}\Rightarrow$$

$$\overset{\substack{\text{πολλαπλασιάζουμε αριθμητή} \\ \text{και παρονομαστή με} \\ \left(\sqrt{1+x}+\sqrt{1-x}\right)\left(\sqrt{2x+1}+1\right)}}{\Rightarrow} \frac{f(x)}{g(x)}=\frac{h(x)\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)\left(\sqrt{2x+1}+1\right)}{\varphi(x)\left(\sqrt{2x+1}-1\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)\left(\sqrt{2x+1}+1\right)}\Rightarrow$$

$$\Rightarrow \frac{f(x)}{g(x)}=\frac{h(x)\left(\sqrt{1+x}^2-\sqrt{1-x}^2\right)\left(\sqrt{2x+1}+1\right)}{\varphi(x)\left(\sqrt{2x+1}^2-1^2\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}\Rightarrow$$

$$\Rightarrow \frac{f(x)}{g(x)}=\frac{h(x)\left(\cancel{1}+x-\cancel{1}+x\right)\left(\sqrt{2x+1}+1\right)}{\varphi(x)\left(2x+\cancel{1}-\cancel{1}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}\Rightarrow$$

$$\Rightarrow \frac{f(x)}{g(x)}=\frac{h(x)\cancel{2x}\left(\sqrt{2x+1}+1\right)}{\varphi(x)\cancel{2x}\left(\sqrt{1+x}+\sqrt{1-x}\right)}\Rightarrow$$

$$\Rightarrow \lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{\lim_{x\to 0}h(x)\lim_{x\to 0}\left(\sqrt{2x+1}+1\right)}{\lim_{x\to 0}\varphi(x)\lim_{x\to 0}\left(\sqrt{1+x}+\sqrt{1-x}\right)}\Rightarrow$$

$$\overset{(1),(2)}{\Rightarrow} \lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{1\cdot\left(\sqrt{2\cdot 0+1}+1\right)}{\frac{1}{2}\left(\sqrt{1+0}+\sqrt{1-0}\right)}=\boxed{2}$$

5.32 6)

Θέτουμε $h(x)=\frac{f(x)}{x^2}$ και $\varphi(x)=\frac{g(x)}{\sigma\upsilon\nu\frac{1}{x}}$

οπότε $\lim_{x\to 0}h(x)=\alpha$ (1) , $\lim_{x\to 0}\varphi(x)=\beta$ (2) , και ακόμη

$$\left.\begin{array}{l} h(x)=\frac{f(x)}{x^2}\Rightarrow f(x)=x^2h(x) \\ \varphi(x)=\frac{g(x)}{\sigma\upsilon\nu\frac{1}{x}}\Rightarrow g(x)=\varphi(x)\sigma\upsilon\nu\frac{1}{x}\end{array}\right\}\Rightarrow f(x)\cdot g(x)=h(x)\varphi(x)x^2\sigma\upsilon\nu\frac{1}{x}\Rightarrow$$

$$\Rightarrow \lim_{x\to 0}f(x)\cdot g(x)=\lim_{x\to 0}h(x)\cdot\lim_{x\to 0}\varphi(x)\cdot\lim_{x\to 0}\left(x^2\sigma\upsilon\nu\frac{1}{x}\right)\quad (3)$$

Όμως

$$-1\le\sigma\upsilon\nu\frac{1}{x}\le 1\overset{x^2\ge 0}{\Rightarrow}-x^2\le x^2\sigma\upsilon\nu\frac{1}{x}\le x^2\quad (4)$$

Όμως $\lim_{x\to 0}\left(-x^2\right)=\lim_{x\to 0}\left(x^2\right)=0$ (5)

Οπότε (4),(5) $\overset{\text{κριτήριο παρεμβολής}}{\Rightarrow}$ $\boxed{\lim_{x\to 0}x^2\sigma\upsilon\nu\frac{1}{x}=0}$ (6)

Έτσι τελικά είναι

$$(3)\overset{(1),(2),(6)}{\Rightarrow}\lim_{x\to 0}\left(f(x)g(x)\right)=\alpha\cdot\beta\cdot 0\Rightarrow\boxed{\lim_{x\to 0}\left(f(x)g(x)\right)=0}$$