5.31 1)

α) Θέτουμε $h(x)=\frac{f(x)-5}{x-2}$ οπότε $\lim_{x\to 2}h(x)=4$ (1) και ακόμη

$$h(x)=\frac{f(x)-5}{x-2}\Rightarrow h(x)(x-2)=f(x)-5\Rightarrow f(x)=h(x)(x-2)+5$$

Οπότε

$$\boxed{\lim_{x\to 2}f(x)}=\lim_{x\to 2}h(x)(x-2)+5=\lim_{x\to 2}h(x)\cdot\lim_{x\to 2}(x-2)+5\overset{(1)}{=}4\cdot 0+5=\boxed{5}$$

β) $$\boxed{\lim_{x\to 2}\frac{f(x)-3x+1}{x^2-4}}\overset{f(x)=h(x)(x-2)+5}{=}\lim_{x\to 2}\frac{h(x)(x-2)+5-3x+1}{x^2-4}=$$

$$=\lim_{x\to 2}\frac{h(x)(x-2)-3x+6}{x^2-4}=\lim_{x\to 2}\frac{h(x)(x-2)-3(x-2)}{(x-2)(x+2)}=$$

$$=\lim_{x\to 2}\frac{\cancel{(x-2)}\left[h(x)-3\right]}{\cancel{(x-2)}(x+2)}\overset{\lim_{x\to 2}h(x)=4}{=}\frac{4-3}{2+2}=\boxed{\frac{1}{4}}$$

5.31 2)

α) Θέτουμε $h(x)=\frac{f(x)-3}{x+1}$ οπότε $\lim_{x\to -1}h(x)=5$ (1) και ακόμη

$$h(x)=\frac{f(x)-3}{x+1}\Rightarrow h(x)(x+1)=f(x)-3\Rightarrow f(x)=h(x)(x+1)+3$$

Οπότε

$$\boxed{\lim_{x\to -1}f(x)}=\lim_{x\to -1}h(x)(x+1)+3=\lim_{x\to -1}h(x)\cdot\lim_{x\to -1}(x+1)+3\overset{(1)}{=}5\cdot 0+3=\boxed{3}$$

β) $$\boxed{\lim_{x\to -1}\frac{f(x)-x-4}{x+1}}\overset{f(x)=h(x)(x+1)+3}{=}\lim_{x\to -1}\frac{h(x)(x+1)+3-x-4}{x+1}=$$

$$=\lim_{x\to -1}\frac{h(x)(x+1)-x-1}{x+1}=\lim_{x\to -1}\frac{h(x)(x+1)-(x+1)}{x+1}$$

$$=\lim_{x\to -1}\frac{\cancel{(x+1)}\left[h(x)-1\right]}{\cancel{x+1}}\overset{\lim_{x\to -1}h(x)=5}{=}5-1=\boxed{4}$$

5.31 3)

α) Θέτουμε $h(x)=\frac{2f(x)+6}{x-4}$ οπότε $\lim_{x\to 4}h(x)=2$ (1) και ακόμη

$$h(x)=\frac{2f(x)+6}{x-4}\Rightarrow h(x)(x-4)=2f(x)+6\Rightarrow 2f(x)=h(x)(x-4)-6\Rightarrow$$

$$\Rightarrow f(x)=\frac{h(x)(x-4)-6}{2}$$

Οπότε

$$\boxed{\lim_{x\to4} f(x)} = \lim_{x\to4}\frac{h(x)(x-4)-6}{2} = \lim_{x\to4}\frac{h(x)\cdot\lim_{x\to4}(x-4)-6}{2} \overset{(1)}{=} \frac{2\cdot0-6}{2} = \boxed{-3}$$

β) $$\boxed{\lim_{x\to4}\frac{f(x)+2x-5}{x-4}} \overset{f(x)=\frac{h(x)(x-4)-6}{2}}{=} \lim_{x\to4}\frac{\frac{h(x)(x-4)-6}{2}+2x-5}{x-4} =$$

$$= \lim_{x\to4}\frac{\frac{h(x)(x-4)-6+4x-10}{2}}{x-4} = \lim_{x\to4}\frac{h(x)(x-4)+4x-16}{2(x-4)} =$$

$$= \lim_{x\to4}\frac{h(x)(x-4)+4(x-4)}{2(x-4)} = \lim_{x\to4}\frac{\cancel{(x-4)}\left[h(x)+4\right]}{2\cancel{(x-4)}} \overset{\lim_{x\to4} h(x)=2}{=} \frac{2+4}{2} = \boxed{3}$$

5.31 4)

α) Θέτουμε $h(x) = \dfrac{f(x)}{\sqrt{x+3}-2}$ οπότε $\lim_{x\to1} h(x) = 8$ (1) και ακόμη

$$h(x) = \frac{f(x)}{\sqrt{x+3}-2} \Rightarrow f(x) = h(x)\left(\sqrt{x+3}-2\right)$$

Οπότε

$$\lim_{x\to1}\boxed{\lim_{x\to1} f(x)} = \lim_{x\to1} h(x)\left(\sqrt{x+3}-2\right) = \lim_{x\to1} h(x)\cdot\lim_{x\to1}\left(\sqrt{x+3}-2\right) \overset{(1)}{=} 8\cdot0 = \boxed{0}$$

β) $$\boxed{\lim_{x\to1}\frac{f(x)}{x-1}} \overset{f(x)=h(x)\left(\sqrt{x+3}-2\right)}{=} \lim_{x\to1}\frac{h(x)\left(\sqrt{x+3}-2\right)}{x-1} \overset{\text{πολλαπλασιάζουμε αριθμητή και παρονομαστή με } \sqrt{x+3}+2}{=}$$

$$= \lim_{x\to1}\frac{h(x)\left(\sqrt{x+3}-2\right)\left(\sqrt{x+3}+2\right)}{(x-1)\left(\sqrt{x+3}+2\right)} = \lim_{x\to1}\frac{h(x)\left(\sqrt{x+3}^{\,2}-2^2\right)}{(x-1)\left(\sqrt{x+3}+2\right)} =$$

$$= \lim_{x\to1}\frac{h(x)(x+3-4)}{(x-1)\left(\sqrt{x+3}+2\right)} = \lim_{x\to1}\frac{h(x)\cancel{(x-1)}}{\cancel{(x-1)}\left(\sqrt{x+3}+2\right)} \overset{(1)}{=}$$

$$= \lim_{x\to1}\frac{h(x)}{\left(\sqrt{x+3}+2\right)} = \frac{8}{\sqrt{1+3}+2} = \boxed{2}$$

5.31 5)

α) Θέτουμε $h(x) = \dfrac{xf(x)-1}{x^2-1}$ οπότε $\lim_{x\to1} h(x) = 2$ (1) και ακόμη

$$h(x) = \frac{xf(x)-1}{x^2-1} \Rightarrow h(x)\left(x^2-1\right) = xf(x)-1 \Rightarrow xf(x) = h(x)\left(x^2-1\right)+1 \Rightarrow$$

$$f(x) = \frac{h(x)\left(x^2-1\right)+1}{x}$$

Οπότε

$$\boxed{\lim_{x\to1} f(x)} = \lim_{x\to1}\frac{h(x)\left(x^2-1\right)+1}{x} = \frac{\lim_{x\to1} h(x)\cdot\lim_{x\to1}\left(x^2-1\right)+1}{\lim_{x\to1} x} \overset{(1)}{=} \frac{2\cdot\left(1^2-1\right)+1}{1} = \boxed{1}$$

β) $\boxed{\lim_{x\to 1}\frac{f(x)-1}{\sqrt{x}-1}} \overset{f(x)=\frac{h(x)(x^2-1)+1}{x}}{=} \lim_{x\to 1}\frac{\frac{h(x)(x^2-1)+1}{x}-1}{\sqrt{x}-1} = \lim_{x\to 1}\frac{\frac{h(x)(x^2-1)+1-x}{x}}{\sqrt{x}-1} =$

$$= \lim_{x\to 1}\frac{h(x)(x^2-1)+1-x}{x(\sqrt{x}-1)} = \lim_{x\to 1}\frac{h(x)(x-1)(x+1)-(x-1)}{x(\sqrt{x}-1)} \overset{\text{πολλαπλασιάζουμε αριθμητή και παρονομαστή με } \sqrt{x}+1}{=}$$

$$= \lim_{x\to 1}\frac{(x-1)[h(x)(x+1)-1](\sqrt{x}+1)}{x(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{(x-1)[h(x)(x+1)-1](\sqrt{x}+1)}{x(\sqrt{x}^2-1^2)} =$$

$$= \lim_{x\to 1}\frac{\cancel{(x-1)}[h(x)(x+1)-1](\sqrt{x}+1)}{x\cancel{(x-1)}} \overset{(1)}{=} \frac{[2(1+1)-1](\sqrt{1}+1)}{1} = \boxed{6}$$