22.16 1)

α) $g(x)=f(x)+x^2 \overset{\text{παραγωγίζουμε}}{\Rightarrow} g'(x)=f'(x)+2x \overset{f'(x)+2x=f(x)+x^2}{\Rightarrow}$

$g'(x)=f(x)+x^2 \overset{g(x)=f(x)+x^2}{\Rightarrow} \boxed{g'(x)=g(x)}$

β) Έχουμε

$g'(x)=g(x)\Rightarrow g(x)=ce^x \Rightarrow f(x)+x^2=ce^x \Rightarrow f(x)=ce^x-x^2$ (1)

Ακόμη

$f(0)=1 \overset{(1)}{\Rightarrow} ce^0-0^2=1\Rightarrow c=1$ (2)

(1),(2) $\Rightarrow \boxed{f(x)=e^x-x^2}$

22.16 2)

α) $g(x)=(x+1)f(x) \overset{\text{παραγωγίζουμε}}{\Rightarrow} g'(x)=f(x)+(x+1)f'(x) \overset{(x+1)f'(x)=xf(x)}{\Rightarrow}$

$\Rightarrow g'(x)=f(x)+xf(x)\Rightarrow g'(x)=(x+1)f(x) \overset{g(x)=(x+1)f(x)}{\Rightarrow} \boxed{g'(x)=g(x)}$

β) Έχουμε

$g'(x)=g(x)\Rightarrow g(x)=ce^x \Rightarrow (x+1)f(x)=ce^x \overset{x\in(0,+\infty)\Rightarrow x\neq -1}{\Rightarrow} f(x)=\dfrac{ce^x}{x+1}$ (1)

Ακόμη

$f(1)=2e \overset{(1)}{\Rightarrow} \dfrac{ce^1}{1+1}=2e\Rightarrow \dfrac{ce}{2}=2e\Rightarrow c=4$ (2)

(1),(2) $\Rightarrow \boxed{f(x)=\dfrac{4e^x}{x+1}}$

22.16 3)

α) $g(x)=\dfrac{f(x)}{\ln x} \overset{\text{παραγωγίζουμε}}{\Rightarrow} g'(x)=\dfrac{f'(x)\ln x-\dfrac{f(x)}{x}}{\ln^2 x} \overset{\substack{(f'(x)-f(x))\ln x=\frac{f(x)}{x}\Rightarrow \\ f'(x)\ln x-f(x)\ln x=\frac{f(x)}{x}\Rightarrow \\ f'(x)\ln x-\frac{f(x)}{x}=f(x)\ln x}}{\Rightarrow}$

$\Rightarrow g'(x)=\dfrac{f(x)\cancel{\ln x}}{\ln^{\cancel{2}} x}\Rightarrow g'(x)=\dfrac{f(x)}{\ln x} \overset{g(x)=\frac{f(x)}{\ln x}}{\Rightarrow} \boxed{g'(x)=g(x)}$

β) Έχουμε

$g'(x)=g(x)\Rightarrow g(x)=ce^x \Rightarrow \dfrac{f(x)}{\ln x}=ce^x \Rightarrow f(x)=ce^x\ln x$ (1)

Ακόμη

$f(e)=e^e \overset{(1)}{\Rightarrow} ce^e\ln e=e^e\Rightarrow ce^e=e^e\Rightarrow c=1$ (2)

(1),(2) $\Rightarrow \boxed{f(x)=e^x\ln x}$

α) $g(x)=\dfrac{f(x)}{x^2+1} \overset{\text{παραγωγίζουμε}}{\Rightarrow} g'(x)=\dfrac{f'(x)(x^2+1)-2xf(x)}{(x^2+1)^2} \overset{f'(x)=\frac{(x+1)^2 f(x)}{x^2+1}}{\Rightarrow}$

$$\Rightarrow g'(x)=\frac{\frac{(x+1)^2 f(x)}{x^2+1}(x^2+1)-2xf(x)}{(x^2+1)^2} \Rightarrow g'(x)=\frac{(x+1)^2 f(x)-2xf(x)}{(x^2+1)^2} \Rightarrow$$

$$\Rightarrow g'(x)=\frac{f(x)(x^2+2x+1-2x)}{(x^2+1)^2} \Rightarrow g'(x)=\frac{f(x)(x^2+1)}{(x^2+1)^2} \Rightarrow$$

$$\Rightarrow g'(x)=\frac{f(x)}{(x^2+1)} \Rightarrow g'(x)=g(x)$$

β) Έχουμε

$$g'(x)=g(x) \Rightarrow g(x)=ce^x \Rightarrow \frac{f(x)}{x^2+1}=ce^x \Rightarrow f(x)=ce^x(x^2+1) \quad (1)$$

Ακόμη

$$f(0)=2 \overset{(1)}{\Rightarrow} ce^0(0^2+1)=2 \Rightarrow c=2 \quad (2)$$

$$(1),(2) \Rightarrow \boxed{f(x)=2e^x(x^2+1)}$$