22.12 1)

α) $$g(x)=\frac{f(x)}{\ln x}\overset{\text{παραγωγίζουμε}}{\Rightarrow}\boxed{g'(x)}=\frac{f'(x)\ln x-f(x)\frac{1}{x}}{\ln^2 x}=\frac{\frac{xf'(x)\ln x-f(x)}{x}}{\ln^2 x}=$$

$$=\frac{xf'(x)\ln x-f(x)}{x\ln^2 x}\overset{xf'(x)\ln x-f(x)=2x^2\ln^2 x}{=}\frac{2x^{\cancel{2}}\cancel{\ln^2 x}}{\cancel{x}\cancel{\ln^2 x}}=\boxed{2x}$$

β) Έχουμε

$$g'(x)=2x\Rightarrow g'(x)=\left(x^2\right)'\Rightarrow g(x)=x^2+c\Rightarrow\frac{f(x)}{\ln x}=x^2+c\Rightarrow$$

$$\Rightarrow f(x)=\left(x^2+c\right)\ln x \quad (1)$$

Ακόμη

$$f(e)=e^2\overset{(1)}{\Rightarrow}\left(e^2+c\right)\ln e=e^2\Rightarrow \cancel{e^2}+c=\cancel{e^2}\Rightarrow c=0 \quad (2)$$

$$(1),(2)\Rightarrow\boxed{f(x)=x^2\ln x}$$

22.12 2)

α) $$g(x)=\frac{f(x)}{e^x}\overset{\text{παραγωγίζουμε}}{\Rightarrow}\boxed{g'(x)}=\frac{e^xf'(x)-e^xf(x)}{e^{2x}}=\frac{\cancel{e^x}\left[f'(x)-f(x)\right]}{e^{\cancel{2x}\,x}}=$$

$$\overset{f'(x)-f(x)=\frac{e^x}{x}}{=}\frac{\frac{\cancel{e^x}\,1}{x}}{\cancel{e^x}}=\boxed{\frac{1}{x}}$$

β) Έχουμε

$$g'(x)=\frac{1}{x}\Rightarrow g'(x)=(\ln x)'\Rightarrow g(x)=\ln x+c\Rightarrow\frac{f(x)}{e^x}=\ln x+c\Rightarrow$$

$$\Rightarrow f(x)=e^x\ln x+ce^x \quad (1)$$

Ακόμη

$$f(1)=e\overset{(1)}{\Rightarrow}e^1\ln 1+ce^1=e\Rightarrow ce=e\Rightarrow c=1 \quad (2)$$

$$(1),(2)\Rightarrow\boxed{f(x)=e^x\ln x+e^x}$$

22.12 3)

α) $$g(x)=e^{-\frac{1}{x}}f(x)\overset{\text{παραγωγίζουμε}}{\Rightarrow}\boxed{g'(x)}=e^{-\frac{1}{x}}\frac{1}{x^2}f(x)+e^{-\frac{1}{x}}f'(x)=$$

$$=e^{-\frac{1}{x}}\left[\frac{1}{x^2}f(x)+f'(x)\right]\overset{\frac{1}{x^2}f(x)+f'(x)=e^{\frac{1}{x}}}{=}e^{-\frac{1}{x}}\cdot e^{\frac{1}{x}}=\boxed{1}$$

β) Έχουμε

$$g'(x)=1\Rightarrow g'(x)=(x)'\Rightarrow g(x)=x+c\Rightarrow e^{-\frac{1}{x}}f(x)=x+c\Rightarrow$$

$$\Rightarrow f(x)=xe^{\frac{1}{x}}+ce^{\frac{1}{x}} \quad (1)$$

Ακόμη

$$f'(x)+\frac{1}{x^2}f(x)=e^{\frac{1}{x}} \overset{x=-1}{\Rightarrow} f'(-1)+\frac{1}{(-1)^2}f(-1)=e^{\frac{1}{-1}} \overset{f'(-1)=0}{\Rightarrow} f(-1)=e^{-1}\Rightarrow$$

$$\overset{(1)}{\Rightarrow}-1\cdot e^{\frac{1}{-1}}+ce^{\frac{1}{-1}}=e^{-1}\Rightarrow -e^{-1}+ce^{-1}=e^{-1}\Rightarrow ce^{-1}=2e^{-1}\Rightarrow c=2 \quad (2)$$

$$(1),(2)\Rightarrow f(x)=xe^{\frac{1}{x}}+2e^{\frac{1}{x}}\Rightarrow \boxed{f(x)=(x+2)e^{\frac{1}{x}}}$$

22.12 4)

α) $$g(x)=\frac{f(x)}{\sqrt{x^2+1}} \overset{\text{παραγωγίζουμε}}{\Rightarrow} \boxed{g'(x)}=\frac{f'(x)\sqrt{x^2+1}-f(x)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{\sqrt{x^2+1}^2}=$$

$$=\frac{\frac{f'(x)\sqrt{x^2+1}^2-xf(x)}{\sqrt{x^2+1}}}{\sqrt{x^2+1}^2}=\frac{f'(x)(x^2+1)-xf(x)}{(x^2+1)\sqrt{x^2+1}} \overset{(x^2+1)f'(x)=xf(x)+e^x(x^2+1)\sqrt{x^2+1}}{=}$$

$$=\frac{\cancel{xf(x)}+e^x(x^2+1)\sqrt{x^2+1}-\cancel{xf(x)}}{(x^2+1)\sqrt{x^2+1}}=\frac{e^x\cancel{(x^2+1)}\cancel{\sqrt{x^2+1}}}{\cancel{(x^2+1)}\cancel{\sqrt{x^2+1}}}=e^x$$

β) Έχουμε

$$g'(x)=e^x\Rightarrow g'(x)=(e^x)'\Rightarrow g(x)=e^x+c\Rightarrow \frac{f(x)}{\sqrt{x^2+1}}=e^x+c\Rightarrow$$

$$\Rightarrow f(x)=(e^x+c)\sqrt{x^2+1} \quad (1)$$

Ακόμη

$$f(0)=1\overset{(1)}{\Rightarrow}(e^0+c)\sqrt{0^2+1}=1\Rightarrow c+1=1\Rightarrow c=0 \quad (2)$$

$$(1),(2)\Rightarrow f(x)=\boxed{e^x\sqrt{x^2+1}}$$